

MODULE - 1

LINEAR ALGEBRA PART-1

MATRICES:-

System of Equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3$$

They can be represented in matrix as

A $\rightarrow$ coefficient of x_1, x_2, x_3 X $\rightarrow$ X variable B $\rightarrow$ Result

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\rightarrow AX = B$$

GAUSS ELIMINATION METHOD:- 3 equations with 3 unknowns

Augmented matrix $\rightarrow$ Row transformations.

The system of equations:-

This method is illustrated by considering a system of 3 independent eqns in 3 unknowns / Square matrix.

The augmented matrix $[A:B]$ is

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & d_1 \\ a_{21} & a_{22} & a_{23} & : & d_2 \\ a_{31} & a_{32} & a_{33} & : & d_3 \end{bmatrix}$$

By applying row transformation continuously, to convert the augmented matrix into Upper triangular matrix, i.e.

$$[A:B] \approx \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' & : & d_1 \\ a_{21}' & a_{22}' & a_{23}' & : & d_2 \\ a_{31}' & a_{32}' & a_{33}' & : & d_3 \end{bmatrix}$$

The given system of linear equations = system of equations

$$\begin{aligned} a_{11}'x_1 + a_{12}'x_2 + a_{13}'x_3 &= d_1 \\ a_{21}'x_1 + a_{22}'x_2 + a_{23}'x_3 &= d_2 \\ a_{31}'x_1 + a_{32}'x_2 + a_{33}'x_3 &= d_3 \end{aligned}$$

is converted as $a_{21}, a_{31}, a_{32} \text{ is } 0$

$$\therefore a_{11}'x_1 + a_{12}'x_2 + a_{13}'x_3 = d_1 \rightarrow \textcircled{1}$$

$$a_{22}'x_2 + a_{23}'x_3 = d_2 \rightarrow \textcircled{2}$$

$$a_{33}'x_3 = d_3 \rightarrow \textcircled{3}$$

Solving $\textcircled{1} \textcircled{2} \textcircled{3}$, we get the values of x_1, x_2, x_3 .

PROBLEMS:

1. Solve by method of Gauss elimination for the system of equations

$$\textcircled{1} \quad x + y + z = 9$$

$$x - 2y + 3z = 8$$

$$2x + y - z = 3$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array} \right] \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}$$

Use of diagonal element

$$R_2 = R_2 - R_1$$

$$R_3 = R_3 - 2R_1$$

$$[A:B] \approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -18+3 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array} \right]$$

$$R_3 = 3R_3 - R_2$$

$$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{array} \right]$$

$$\begin{aligned} R_3 &= 3R_3 - R_2 \\ &= 3(-1) - (-3) \\ &= 3 + 3 = 0 \\ &= 3(-3) - (2) \\ &= -11 \\ &= 3(-15) - (-1) \\ &= -45 + 1 = -44 \end{aligned}$$

$$x + y + z = 9 \rightarrow (1)$$

$$-3y + 2z = -1 \rightarrow (2)$$

$$-11z = -44 \rightarrow (3)$$

From (3)

$$-11z = -44$$

$$z = \frac{-44}{-11}$$

$$\underline{z = 4}$$

Substitute $z = 4$ in (2)

$$-3y + 2(4) = -1$$

$$-3y + 8 = -1$$

$$-3y = -9$$

$$y = \frac{-9}{-3}$$

$$\underline{y = 3}$$

Substitute $z = 4$ & $y = 3$ in (1)

$$x + 3 + 4 = 9$$

$$x = 9 - 7$$

$$\underline{x = 2}$$

Thus the solutions are $x = 2$, $y = 3$ & $z = 4$

(11)

$$2x + y + 4z = 12$$

$$4x + 11y - z = 33$$

$$8x - 3y + 2z = 20$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 4 & 11 & -1 & 33 \\ 8 & -3 & 2 & 20 \end{array} \right]$$

$$R_2 = R_2 - 2R_1, R_3 = R_3 - 4R_1$$

$$[A|B] \approx \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & -7 & -14 & -28 \end{bmatrix}$$

$$R_3 = 7R_2 + 9R_3 \text{ or } 9R_3 + 7R_2$$

$$[A|B] \approx \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & 0 & -189 & -189 \end{bmatrix}$$

$$9(-14) + 7(-9) = -126 - 63 = -189$$

$$9(-28) + 7(9) = -252 + 63 = -189$$

$$2x + y + 4z = 12 \rightarrow \textcircled{1}$$

$$9y - 9z = 9 \rightarrow \textcircled{2}$$

$$-189z = -189 \rightarrow \textcircled{3}$$

From $\textcircled{3}$

$$-189z = -189$$

$$z = \frac{-189}{-189}$$

$$z = 1$$

$$\underline{\underline{z = 1}}$$

Sub $z = 1$ in $\textcircled{2}$

$$9y - 9(1) = 9$$

$$9y = 18$$

$$y = \frac{18}{9}$$

$$\underline{\underline{y = 2}}$$

Sub $z = 1$ & $y = 2$ in $\textcircled{1}$

$$2x + 2 + 4(1) = 12$$

$$2x + 6 = 12$$

$$2x = 6$$

$$x = \frac{6}{2}$$

$$\underline{\underline{x = 3}}$$

$\therefore$ Thus the solution is $x = 3, y = 2, z = 1$

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$$5x_1 + x_2 + x_3 + x_4 = 4$$

$$x_1 + 7x_2 + x_3 + x_4 = 12$$

$$x_1 + x_2 + 6x_3 + x_4 = -5$$

$$x_1 + x_2 + x_3 + 4x_4 = -6$$

Augmented matrix

$$[A:B] = \begin{bmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 12 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 1 & 1 & 1 & 1 & : & -6 \end{bmatrix}$$

$$R_2 = 5R_2 - R_1$$

$$R_3 = 5R_3 - R_1$$

$$R_4 = 5R_4 - R_1$$

$$[A:B] \approx \begin{bmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 4 & : & 56 \\ 0 & 4 & 29 & 4 & : & -29 \\ 0 & 4 & 4 & 19 & : & -34 \end{bmatrix}$$

$$R_3 = 34R_3 - 4R_2$$

$$R_4 = 34R_4 - 4R_2$$

$$[A:B] \approx \begin{bmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 4 & : & 56 \\ 0 & 0 & 970 & 120 & : & -1210 \\ 0 & 0 & 970 & 630 & : & -1380 \end{bmatrix}$$

$$R_4 = 970R_4 - 120R_3$$

$$[A:B] \approx \begin{bmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 0 & 34 & 4 & 4 & : & 56 \\ 0 & 0 & 970 & 120 & : & -1210 \\ 0 & 0 & 0 & 596700 & : & -1193400 \end{bmatrix}$$

$$5x_1 + x_2 + x_3 + x_4 = 4 \rightarrow \textcircled{1}$$

$$34x_2 + 4x_3 + 4x_4 = 56 \rightarrow \textcircled{2}$$

$$970x_3 + 120x_4 = -1210 \rightarrow \textcircled{3}$$

$$596700x_4 = -1193400 \rightarrow \textcircled{4}$$

from $\textcircled{4}$

$$596700x_4 = -1193400$$

$$x_4 = \frac{-1193400}{596700}$$

$$\underline{\underline{x_4 = -2}}$$

sub in $\textcircled{3}$

$$970x_3 + 120(-2) = -1210$$

$$970x_3 - 240 = -1210$$

$$x_3 = \frac{-970}{970}$$

$$\underline{\underline{x_3 = -1}}$$

sub in $\textcircled{2}$

$$34x_2 + 4(-1) + 4(-2) = 56$$

$$34x_2 - 4 - 8 = 56$$

$$34x_2 - 12 = 56$$

$$34x_2 = 56 + 12$$

$$x_2 = \frac{68}{34}$$

$$\underline{\underline{x_2 = 2}}$$

sub in $\textcircled{1}$

$$5x_1 + 2 + (-1) + (-2) = 4$$

$$5x_1 + 2 - 1 - 2 = 4$$

$$5x_1 - 1 = 4$$

$$5x_1 = 5$$

$$x_1 = \frac{5}{5}$$

$$\underline{\underline{x_1 = 1}}$$

Thus the solution is $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$

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$$x_1 - 2x_2 + 3x_3 = 2$$

$$3x_1 - x_2 + 4x_3 = 4$$

$$2x_1 + x_2 - 2x_3 = 5$$

The augmented matrix is.

$$[A:B] \approx \left[\begin{array}{ccc|c} \textcircled{1} & -2 & 3 & 2 \\ 3 & -1 & 4 & 4 \\ 2 & 1 & -2 & 5 \end{array} \right]$$

$$R_2 = R_2 - 3R_1$$

$$R_3 = R_3 - 2R_1$$

$$[A:B] \approx \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & \textcircled{5} & -5 & -2 \\ 0 & 5 & -8 & 1 \end{array} \right]$$

$$R_3 = R_3 - R_2$$

$$[A:B] \approx \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -3 & 3 \end{array} \right]$$

$$x_1 - 2x_2 + 3x_3 = 2 \rightarrow \textcircled{1}$$

$$5x_2 - 5x_3 = -2 \rightarrow \textcircled{2}$$

$$-3x_3 = 3 \rightarrow \textcircled{3}$$

From $\textcircled{3}$

$$-3x_3 = 3$$

$$x_3 = \frac{3}{-3}$$

$$\underline{\underline{x_3 = -1}}$$

Sub. $x_3 = -1$ in $\textcircled{2}$

$$5x_2 - 5(-1) = -2$$

$$5x_2 + 5 = -2$$

$$5x_2 = -2 - 5$$

$$x_2 = \frac{-7}{5}$$

$$\underline{\underline{x_2 = -1.4}}$$

Sub $x_3 = -1$ & $x_2 = -1.4$ in $\textcircled{1}$

$$x_1 - 2(-1.4) + 3(-1) = 2$$

$$x_1 + 2.8 - 3 = 2$$

$$x_1 - 0.2 = 2$$

$$\underline{\underline{x_1 = 2.2}}$$

Thus the solution is $x_1 = 2.2$, $x_2 = -1.4$, $x_3 = -1$

$$\textcircled{1} \quad 2x + 5y + 7z = 52$$

$$2x + y - z = 0$$

$$x + y + z = 9$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & 9 \end{array} \right]$$

$$R_3 \leftrightarrow R_1 \text{ (Interchange } R_3 \text{ \& } R_1)$$

$\therefore$

$$[A:B] \approx \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 9 \\ 2 & 1 & -1 & 0 \\ 2 & 5 & 7 & 52 \end{array} \right]$$

$$R_2 = R_2 - 2R_1 \quad R_3 = R_3 - 2R_1$$

$$[A:B] \approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & \textcircled{-1} & -3 & -18 \\ 0 & 3 & 5 & 34 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$[A:B] \approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{array} \right]$$

$$x + y + z = 9 \rightarrow \textcircled{1}$$

$$-y - 3z = -18 \rightarrow \textcircled{2}$$

$$-4z = -20 \rightarrow \textcircled{3}$$

From (3)

$$-4z = -20$$

$$z = \frac{-20}{-4}$$

$$z = 5$$

$$\underline{\underline{z = 5}}$$

Sub $z = 5$ in (2)

$$-y - 3(5) = -18$$

$$-y - 15 = -18$$

$$-y = -18 + 15$$

$$-y = -3$$

$$y = 3$$

$$\underline{\underline{y = 3}}$$

Sub $z = 5$ & $y = 3$ in (1)

$$x + 3 + 5 = 9$$

$$x + 8 = 9$$

$$x = 9 - 8$$

$$x = 1$$

$$\underline{\underline{x = 1}}$$

$\therefore$ The solution is $x = 1, y = 3, z = 5$

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$$x_1 + x_2 + x_3 + x_4 = 2$$

$$2x_1 - x_2 + 2x_3 - x_4 = -5$$

$$3x_1 + 2x_2 + 3x_3 + 4x_4 = 7$$

$$x_1 - 2x_2 - 3x_3 + 2x_4 = 5$$

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -1 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1, R_3 = R_3 - 3R_1, R_4 = R_4 - R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{bmatrix}$$

$$R_3 = R_3 + \frac{1}{3}R_2, R_4 = R_4 - R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & 4 & 12 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$[A|B] \approx \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & 0 & -4 & 4 & 12 \\ 0 & 0 & 0 & 2 & 4 \end{array} \right]$$

$$x_1 + x_2 + x_3 + x_4 = 2 \rightarrow \textcircled{1}$$

$$-3x_2 + 0x_3 - 3x_4 = -9 \rightarrow \textcircled{2}$$

$$-4x_3 + 4x_4 = 12 \rightarrow \textcircled{3}$$

$$2x_4 = 4 \rightarrow \textcircled{4}$$

From $\textcircled{4}$

$$2x_4 = 4$$

$$x_4 = \frac{4}{2}$$

$$\underline{x_4 = 2}$$

Sub $x_4 = 2$ in $\textcircled{3}$

$$-4x_3 + 4x_4 = 12$$

$$-4x_3 + 4(2) = 12$$

$$-4x_3 + 8 = 12$$

$$-4x_3 = 12 - 8$$

$$-4x_3 = 4$$

$$x_3 = \frac{-4}{-4}$$

$$\underline{x_3 = 1}$$

Sub $x_3 = -1$ & $x_4 = 2$ in $\textcircled{2}$

$$-3x_2 + 0 - 3(2) = -9$$

$$-3x_2 - 6 = -9$$

$$-3x_2 = -9 + 6$$

$$x_2 = \frac{-3}{-3}$$

$$\underline{x_2 = 1}$$

$$\underline{x_3 = -1}$$

Sub $x_2 = 1, x_3 = -1, x_4 = 2$ in $\textcircled{1}$

$$x_1 + 1 + (-1) + 2 = 2$$

$$x_1 + 1 - 1 + 2 = 2$$

$$x_1 = 2 - 2$$

$$\underline{x_1 = 0}$$

Thus the solution is $x_1 = 0, x_2 = 1, x_3 = -1, x_4 = 2$

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$$0.2x + 0.3y - 0.1z = 0.5$$

$$0.4x + 0.4y - 0.3z = 0.3$$

$$0.2x - 0.3y + 0.2z = 0.2$$

The coefficients of x is made to 1. by dividing with 0.2, 0.4 & 0.2

$$0.2x + 0.3y - 0.1z = 0.5$$

$$0.4x + 0.4y - 0.3z = 0.3$$

$$0.2x - 0.3y + 0.2z = 0.2$$

$$\Rightarrow x + 1.5y - 0.5z = 2.5 \rightarrow \textcircled{1}$$

$$x + y - 0.75z = 0.75 \rightarrow \textcircled{2}$$

$$x - 1.5y + z = 1 \rightarrow \textcircled{3}$$

Subtraction of $\textcircled{2}$ & $\textcircled{3}$ from $\textcircled{1}$.

$$x + 1.5y - 0.5z = 2.5$$

$$-0.5y + 0.25z = 1.75$$

$$3y - 1.5z = 1.5$$

y is made to 1 by dividing with 0.5

$$x + 1.5y - 0.5z = 2.5$$

$$y + 0.5z = 3.5$$

$$y - 0.5z = 0.5$$

Subtract $\textcircled{3}$ from $\textcircled{2}$

$$x + 1.5y - 0.5z = 2.5$$

$$y + 0.5z = 3.5$$

$$\underline{\underline{z = 3}}$$

* :- +, -, x, ÷

Additive identity - 1
Multiplicative - 1

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Sub $z = 3$ in (2)

$$y + 0.5(3) = 3.5$$

$$y + 1.5 = 3.5$$

$$\underline{y = 2}$$

Sub $y = 2$ & $z = 3$ in (1)

$$x + 1.5(2) - 0.5(3) = 2.5$$

$$x + 3 - 1.5 = 2.5$$

$$x + 1.5 = 2.5$$

$$x = 2.5 - 1.5$$

$$\underline{x = 1}$$

∴ The solution is $x = 1, y = 2$ & $z = 3$

VECTOR SPACES.

Group :- A non empty set equipped with 1 binary operation $* [+, -, ÷, \times]$ is called group if it satisfies the following axioms/postulates.

∃ → There exists

∀ → for all

① Closure property :- $\forall a, b \in G \text{ [set]} \Rightarrow a * b \in G$. $\in$ → belongs to

② Associative property :- $\forall a, b, c \in G \Rightarrow a * (b * c) = (a * b) * c$

③ Existence of Identity :- There exists an element $e \in G$ such that $a * e = a = e * a$. The element e is called the identity

④ Existence of Inverse :- Each element of G posses a Inverse, that means $\forall a \in G \rightarrow$ There exists an element $b \in G$ such that

$$a * b = e = b * a$$

mult

add

Then b is called Inverse of a & we write it as $b = a^{-1}$

$$a * a^{-1} = e = a^{-1} * a$$

↓
Inverse

Then the set G is said to be the group

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And also, a group with commutative property is known as Abelian group (or) commutative group i.e.

$$\forall a, b \in G \Rightarrow a * b = b * a$$

$\therefore$ '*' is commutative on G .

Example:- (i) set of integers under addition is an abelian group as $(\mathbb{Z}, +)$ is a group & also $\forall a, b \in \mathbb{Z}$.

(ii) set of rational no^r under multiplication, real no^r $\mathbb{R}$ under multiplication, complex noⁿ under multiplication

$$(\mathbb{Q}, \cdot), (\mathbb{R}, \cdot), (\mathbb{C}, \cdot) \xrightarrow{\text{Multiplication}}$$

Rings:- Suppose ' R ' is a non empty set equipped with 2 binary operations called addition & multiplication & denoted by '+' & ' $\cdot$ ' respectively.

Then the algebraic structure $(R, +, \cdot)$ is known as ring if the following axioms are satisfied

(i) $(R, +)$ is an abelian group

(i) closure property for '+': $\forall a, b \in G \Rightarrow a + b \in G$

(ii) Associative property for '+': $\forall a, b, c \in G \Rightarrow (a + b) + c = (a + (b + c))$

(iii) Existence of identity '+': $\forall a \in R \exists 0 \in R$ such that (i)

$$a + 0 = 0 + a = a$$

$\therefore 0$ is known as additive identity

(iv) Existence of inverse '+': $\forall a \in R \exists a^{-1} \in R$ or

$$\forall a \in R \exists -a \in R \text{ such that } a + (-a)$$

$$= a - a$$

$$= 0$$

$\therefore -a$ is known as additive inverse of a

(2) $(R, \cdot)$ is a semigroup.

(i) Closure property for ' $\cdot$ ' :- $\forall a, b \in R \Rightarrow a \cdot b \in R$.

(ii) Associative property for ' $\cdot$ ' :- $\forall a, b, c \in R \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$
 $\hookrightarrow$ semigroup.

(3) Multiplication ' $\cdot$ ' distributes over addition '+' from left & also from right

$$\forall a, b, c \in R$$

$$(i) a \cdot (b + c) = (a \cdot b) + a \cdot c$$

$$(ii) a + (b \cdot c) = \cancel{a \cdot b} + b \cdot c$$

$$(ii) (a + b) \cdot c = a \cdot c + b \cdot c$$

Example :- $(\mathbb{I}, +, \cdot)$ is a ring

* Commutative Ring :- Ring $(R, +, \cdot)$ is a commutative ring if $\forall a, b \in R \Rightarrow a \cdot b = b \cdot a$

Example :- $(\mathbb{I}, +, \cdot)$

* Ring with Unity :- Ring $(R, +, \cdot)$ is known as ring with unity $\forall a \in R \exists 1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$

(*) Fields :- A commutative ring with unity in which every non-zero element possesses their multiplicative inverse is known as field. i.e. $(F, +, \cdot)$ is a field if

(i) $(F, +)$ is an abelian group

(ii) $(F, \cdot)$ is an abelian group

(iii) Multiplication distributes over addition from the left and also from the right

Ex :- $(\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot), (\mathbb{C}, +, \cdot)$ are examples of fields
 $\downarrow$
 Real no

NOTE: If $(F, +, \cdot)$ is a field then F contains atleast 2 elements, 0 element & unit element i.e. additive identity element & multiplicative identity element.

VECTOR SPACE

A vector space is abstract algebraic entities. Vector in the space form a linear combinations of finitely many vectors and scalars.

A vector space or a linear space is a group of objects called vectors, added collectively & multiplied by a number called scalars.

- Scalars are usually considered to be real numbers, but there are few cases of scalar multiplication by rational no's, complex numbers etc with vector spaces.

Define vector space with an example

Let 'F' be a field & 'V' be a non empty set, the non empty set V is said to be a vector space over field F if it satisfies the following axioms.

2) V is an abelian group with respect to addition.

→ Closure law - $\forall \alpha, \beta \in V \Rightarrow \alpha + \beta \in V$

→ Associative Law - $\forall \alpha, \beta, \gamma \in V \Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

→ Identity law: $\forall \alpha \in \beta. \rightarrow \alpha + 0 = \alpha$

→ Inverse: $\forall a \in V \Rightarrow a + (-a) = 0$

→ Commutative law - $\forall a, b \in C \Rightarrow a+b = b+a$

closed with respect to scalar multiplication

$$\forall a \in V, \forall \alpha \in F \Rightarrow \alpha \cdot a \in V$$

$\downarrow$ vectors $\downarrow$ field

$n \rightarrow n$ tuples

$2 \rightarrow 2$ tuples.

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(iii) With addition & multiplication, it should satisfy the following axioms.

$$\rightarrow \alpha(a+b) = \alpha a + \alpha b \quad \forall \alpha \in F \text{ \& } a, b \in V$$

$$\rightarrow (\alpha_1 + \alpha_2)a = \alpha_1 a + \alpha_2 a \quad \forall \alpha_1, \alpha_2 \in F \text{ \& } a \in V$$

$$\rightarrow (\alpha_1 \alpha_2)a = \alpha_1(\alpha_2 a) \quad \forall \alpha_1, \alpha_2 \in F \text{ \& } a \in V$$

$$\rightarrow 1 \cdot a = a \quad \forall 1 \in F \text{ \& } a \in V$$

NOTE:- Elements of vectors ^{space} V are called vectors & the element of field of F are called scalars.

Vectors in R^n

The set of all ordered triples (a, b, c) of real nos is called Euclidean three space (3 space) & is denoted by R^3 . And the set of all n tuples of real numbers denoted by R^n is called Euclidean n -space.

Examples:- The ordered pair $(2, -3)$ belongs to R^2 . It is a 2 tuple of dimensions 2.

The ordered triple $(6, 7, -8)$ belongs to R^3 . It is a 3 tuple of dimensions 3.

Examples of Vector space:-

(i) ~~Let~~ Prove that the set of all vectors in a plane over the field of real nos is a vector space with respect to vector addition & scalar multiplication.

' R ' $\rightarrow$ real

Let V denote the set of all vectors & ' R ' be the field of real numbers.

$\therefore$ The element of V are the ordered pairs (x, y)

where $(x, y) \in R$

$$\therefore V = \{(x, y) : x, y \in R\}. \quad R = \{0, \pm 1, \pm 1/2, \dots\}.$$

(i) $(V, +)$ is an abelian group.

(a) Closure law: $\forall (x, y) \in V \rightarrow x + y \in V$

(b) Associative law: $\forall a, b, c \in V \rightarrow a + (b + c) = (a + b) + c$

(c) Identity law: $\forall a \in V, 0 \in V \rightarrow a + 0 = 0 + a = a$

(d) Inverse law: $\forall a \in V \rightarrow a + (-a) = 0 \in V$ (or)

$$\forall a \in V \exists -a \in V \rightarrow a + (-a) = 0.$$

(e) Commutative law: $\forall a, b \in V$ such that $a + b = b + a$

(ii) Scalar multiplication in V

(a) For $(a, b) \in V$ & $\alpha \in R$ then $\alpha(a + b) = \alpha a + \alpha b$

(b) For $a \in V$ & $\alpha, \beta \in R$ then $(\alpha + \beta)a = \alpha a + \beta a$

(c) For $a \in V$ & $\alpha, \beta \in R$ then $\alpha(\beta a) = (\alpha\beta)a$

(d) For $a \in V$ & $1 \in R$ then $1 \cdot a = a$

Thus the set V satisfies all the properties of ^{vector} addition & scalar multiplication.

$\therefore V$ is a vector space.

⑥ Prove that the set C of all complex numbers (the set of all ordered pairs of real nos) is a vector space over field R of all real nos, where vector addition is defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \quad \forall x_1, x_2, y_1, y_2 \in R$ & scalar multiplication is defined by $d(x_1, x_2) = (dx_1, dx_2) \quad \forall d \in R$.

8) we observe that 'C' is closed under vector addition & scalar multiplication

(i) $(C, +)$ is an abelian group

→ Closure Property:- $\forall (x_1, x_2), (y_1, y_2) \in C$, then
$$[(x_1, x_2) + (y_1, y_2)] = [(x_1 + y_1), (x_2 + y_2)]$$
$$= [(y_1, y_2) + (x_1, x_2)] \in C$$

→ Associative law:- $\forall (x_1, x_2), (y_1, y_2), (z_1, z_2) \in C$ then
$$[(x_1, x_2) + [(y_1, y_2) + (z_1, z_2)]]$$
$$= [(x_1, x_2) + (y_1 + z_1, y_2 + z_2)]$$
$$= (x_1 + y_1 + z_1, x_2 + y_2 + z_2)$$
$$= (x_1 + y_1, x_2 + y_2) + (z_1, z_2)$$
$$= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2)$$

→ Identity Law:- $\forall x_1, x_2 \in C \exists (0, 0) \in C$ such that
$$(x_1, x_2) + (0, 0) = (x_1 + 0, x_2 + 0)$$
$$= \underline{(x_1, x_2)}$$

$$\therefore (0, 0) + (x_1, x_2) = (0 + x_1, 0 + x_2)$$
$$= \underline{(x_1, x_2)}$$

→ Inverse Law:- for any $(x_1, x_2) \in C \exists (-x_1, -x_2) \in C$ such that
$$(x_1, x_2) + (-x_1, -x_2) = [(x_1 - x_1), (x_2 - x_2)]$$
$$= \underline{(0, 0)}$$

$$\therefore (-x_1, -x_2) + (x_1, x_2) = [(-x_1 + x_1), (-x_2 + x_2)]$$
$$= \underline{(0, 0)}$$

→ Commutative law: $\forall (x_1, x_2), (y_1, y_2) \in C$ then

$$[(x_1, x_2) + (y_1, y_2)] = [(x_1 + y_1), (x_2 + y_2)]$$

$$= [(y_1 + x_1), (y_2 + x_2)]$$

$$[(x_1, x_2) + (y_1, y_2)] = [(y_1, y_2) + (x_1, x_2)]$$

$\therefore$ Complex not under addition is an abelian group

(ii) Scalar Multiplication on C

$\forall \alpha$ (scalar)

$$\Rightarrow \alpha [(x_1, x_2) + (y_1, y_2)] = \alpha [(x_1 + y_1), (x_2 + y_2)]$$

$$= [\alpha(x_1 + y_1), \alpha(x_2 + y_2)]$$

$$\because d(x_1, x_2) = (d x_1, d x_2)$$

$$= [\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2]$$

$$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$$

$$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$$

$$\therefore \alpha [(x_1, x_2) + (y_1, y_2)] =$$

$$\alpha(x_1, x_2) + \alpha(y_1, y_2)$$

$$\Rightarrow (a+b)(x_1, x_2) = [(a+b)(x_1), (a+b)(x_2)]$$

$$= [ax_1 + bx_1, ax_2 + bx_2]$$

$$= [(ax_1, ax_2) + (bx_1, bx_2)]$$

$$= [a(x_1, x_2) + b(x_1, x_2)]$$

$$\therefore (a+b)(x_1, x_2) = a(x_1, x_2) + b(x_1, x_2)$$

$$\Rightarrow a[b(x_1, x_2)] = a[bx_1, bx_2]$$

$$= (abx_1, abx_2)$$

$$a[b(x_1, x_2)] = ab(x_1, x_2)$$

$$\Rightarrow 1 \cdot (x_1, x_2) = (x_1, x_2) \quad \forall (x_1, x_2) \in C$$

classmate Thus C is a vector space over $\mathbb{R}$.

- (3) Show that the set V of all real valued continuous function $f: x$ defined on interval $[0, 1]$ is a vector space over the field R of real numbers w.r.t vector addition & scalar multiplication defined by $(f_1 + f_2)x = f_1(x) + f_2(x)$
 $\forall f_1, f_2 \in V, (\alpha f_1)x = \alpha f_1(x) \quad \forall \alpha \in R, f_1 \in V$.

So (i) $(V, +)$ is an abelian group

→ Closure law:- $(f_1 + f_2)x$

$\forall f_1, f_2 \in V$, then

$$(f_1 + f_2)x = f_1(x) + f_2(x)$$

→ Associative law:- $\forall f_1, f_2, f_3 \in V$ then

$$\begin{aligned} [(f_1 + f_2) + f_3]x &= (f_1 + f_2)(x) + f_3(x) \\ &= [f_1(x) + f_2(x)] + f_3(x) \\ &= f_1(x) + [f_2(x) + f_3(x)] \\ &= f_1(x) + [(f_2 + f_3)x] \\ &= [f_1 + (f_2 + f_3)]x \end{aligned}$$

$$\Rightarrow [(f_1 + f_2) + f_3]x = [f_1 + (f_2 + f_3)]x$$

→ Identity law:- Define a function 0 such that $0x \Rightarrow 0(x) = 0$
 $\forall x \in [0, 1]$

$$\begin{aligned} \therefore (0 + f_1)(x) &= 0(x) + f_1(x) = 0 + f_1(x) = f_1(x) \\ (f_1 + 0)(x) &= f_1(x) + 0(x) = f_1(x) + 0 = f_1(x) \end{aligned}$$

Thus 0 is the additive identity in V

→ Existence of Inverse:- For a function f , the function $-f$ is defined by

$$(-f)(x) = -f(x) \text{ is called additive inverse}$$

i.e. $[f + (-f)](x) = f(x) + (-f)(x) =$
 $f(x) - f(x) = 0 = 0(x)$

$$\text{Hence } (f - f)(x) = 0(x)$$

→ Commutative law:- Let $f, g \in V$ then -

$$(f + g)(x) = f(x) + g(x) \text{ by definition}$$

then

$$g(x) + f(x) = (g + f)(x)$$

$\therefore V$ is an abelian group under addition

(ii) Scalar multiplication on V

⇒ For $f, g \in V$ & $\alpha \in \mathbb{R}$ then

$$[\alpha(f + g)](x) = [\alpha(f + g)](x)$$

$$= \alpha[f(x) + g(x)]$$

$$= \alpha f(x) + \alpha g(x)$$

$$= (\alpha f)(x) + (\alpha g)(x)$$

$$= (\alpha f + \alpha g)(x)$$

⇒ For $f \in V$ & $\alpha, \beta \in \mathbb{R}$ then

$$[(\alpha + \beta)f](x) = (\alpha + \beta)f(x)$$

$$= \alpha f(x) + \beta f(x)$$

$$= (\alpha f)(x) + (\beta f)(x)$$

$$= (\alpha f + \beta f)(x)$$

$$\Rightarrow (\alpha + \beta)f = \alpha f + \beta f$$

$\Rightarrow$ for $\delta_i \in V$ & $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} [\alpha(\beta\delta_i)x] &= \alpha((\beta\delta_i)x) \\ &= \alpha[\beta\delta_i(x)] \\ &= \alpha\beta\delta_i(x) \end{aligned}$$

$$\Rightarrow \underline{[\alpha(\beta\delta_i)(x)] = \alpha\beta\delta_i(x)}$$

$\Rightarrow$ for $\delta_i \in V$ & $1 \in \mathbb{R}$

$$\cancel{1\delta_i(x)} \quad (1\delta_i)(x) = 1\delta_i(x) = \delta_i(x)$$

$\therefore V$ satisfies all the properties of vector space
Hence V is the vector space of V

Subspace:- A non empty subset 'w' of a vector space V over a field F is called a subspace of V , if w is itself a vector space over F under the same operation of addition & scalar multiplication as defined in V

Example:- (1) The set $\{0\}$ consisting of 0 vector of V is a subspace

(2) The whole vector space V is a subspace of V
These 2 subspaces are called trivial or improper subspaces of V . Any subspace $w(V)$ different from $\{0\}$ (set zero) & V is called a proper subspace of V .

Theorem 1 :- A non empty subset $W(V)$ is a subspace of V if & only if (iff) for each pair of vectors α, β in W & each scalar 'c' in field F then the vector $c\alpha + \beta$ is again in W .

Proof :- Suppose 'W' is a non empty subset of V such that $c\alpha + \beta$ belongs to $W \forall \alpha, \beta \in W$ & $c \in F$

Since W is a non empty, there is a vector ' α ' in W & hence
(-1) $\alpha + \alpha = 0 \in W$ [Additive Identity]

In particular (-1) $\alpha = -\alpha \in W$

$\hookrightarrow$ multiplicative identity (additive inverse)

Finally if α & β are in W then
 $\alpha + \beta = \beta + \alpha \in W$

Thus W is a subspace of V

Conversely :- If W is a subspace of V
 $\alpha, \beta \in W$ & $c \in F$ then
 $c\alpha + \beta \in W$

Theorem 2 :- The intersections of any 2 subspaces of a vector space V over field F is also a subspace of V

Proof :- Let S & T be any 2 subspaces of the vector space V over field F

Now consider $S \cap T = \{\alpha \mid \alpha \in S \text{ & } \alpha \in T\}$

Since S & T are subspaces, $0 \in S$ & $0 \in T$

$\therefore 0 \in S \cap T$ & hence $S \cap T \neq \emptyset$ (nonempty).

We shall show that $\forall \alpha, \beta \in S \cap T$

$$c_1\alpha + c_2\beta \in S \cap T \quad \forall c_1, c_2 \in F$$

Now consider $\alpha, \beta \in S \cap T$

$$\Rightarrow \alpha, \beta \in S \text{ & } \alpha, \beta \in T$$

3 → Triplets

1 → such that

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$$\Rightarrow C_1\alpha + C_2\beta \in S, \& C_1\alpha + C_2\beta \in T$$

$$\forall C_1, \& C_2 \in S \cap T$$

$$\Rightarrow C_1\alpha + C_2\beta \in S \cap T.$$

Hence $S \cap T$ is a subspace of V .

4. Let $V = \mathbb{R}^3$, the vector space of all ordered triplets of real numbers over field F of real no's. Show that the subset $W = \{(x_1, 0, 0) : x_1 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Sol The element $0 \in (0, 0, 0) \in W$ thus W is a non empty set

Let $\alpha_1 = (x_1, 0, 0)$ &

$\alpha_2 = (x_2, 0, 0)$ be any 2 elements of W .

Then $\alpha_1 + \alpha_2 = (x_1, 0, 0) + (x_2, 0, 0)$

$$= (x_1 + x_2, 0 + 0, 0 + 0)$$

$$= (x_1 + x_2, 0, 0) \in W$$

$\therefore W$ is closed under addition

Again for any scalar $c \in \mathbb{R}$

$$c\alpha_1 = c(x_1, 0, 0)$$

$$= (cx_1, 0, 0) \in W$$

$\therefore W$ is closed under multiplication

Hence W is a subspace of $\mathbb{R}^3$

5. Show that the subset $W = \{(x_1, x_2, x_3) : (such that) x_1 + x_2 + x_3 = 0\}$ of the vector space $V_3(\mathbb{R})$ is a subspace of $V_3(\mathbb{R})$.

Sol Let $\alpha = (x_1, x_2, x_3)$ & $\beta = (y_1, y_2, y_3) \in W$

$$\therefore x_1 + x_2 + x_3 = 0 \& y_1 + y_2 + y_3 = 0 \rightarrow \textcircled{1}$$

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Now consider

$$\begin{aligned} C_1\alpha + C_2\beta &= C_1(x_1, x_2, x_3) + C_2(y_1, y_2, y_3) \\ &= (C_1x_1, C_1x_2, C_1x_3) + (C_2y_1, C_2y_2, C_2y_3) \\ &= (C_1x_1 + C_2y_1, C_1x_2 + C_2y_2, C_1x_3 + C_2y_3) \end{aligned}$$

To show that $C_1\alpha + C_2\beta \in W$.

we have to show that sum of the components of $C_1\alpha + C_2\beta$ is 0

$$\begin{aligned} \text{Consider } (C_1x_1 + C_2y_1, C_1x_2 + C_2y_2, C_1x_3 + C_2y_3) \\ &= C_1(x_1 + x_2 + x_3) + C_2(y_1 + y_2 + y_3) \\ &= C_1(0) + C_2(0) \\ &= 0 \end{aligned}$$

$$\therefore C_1\alpha + C_2\beta = 0 \in W$$

$\therefore W$ is a subspace of $V_3(\mathbb{R})$

LINEAR COMBINATION:

Let V be a vector space over field F , and let $v_1, v_2, v_3, \dots, v_n$ belongs to V ($\in V$). Any vector of the form $a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n \in V$ where $a_i \in F$ is called a linear combinations of $v_1, v_2, v_3, \dots, v_n$.

LINEAR DEPENDENCE

Let V be a vector space over field F . The vectors $v_1, v_2, v_3, \dots, v_n$ are said to be linearly dependent over field F if there exists (3) a scalar $a_1, a_2, a_3, \dots, a_n \in F$ not all zero but linear combinations is 0. i.e. $a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n = 0$.

But all $a_i \neq 0$ where $i \in \mathbb{N}$.

Example:- The set $S = \{(1, 0, 1), (1, 1, 0), (-1, 0, -1)\}$ is linearly dependent in $\mathbb{R}^3(\mathbb{R})$.

Sol

As possible let us consider

$$a_1(1, 0, 1) + a_2(1, 1, 0) + a_3(-1, 0, -1) = (0, 0, 0)$$

$$\Rightarrow a_1 + a_2 - a_3 = 0 \rightarrow \textcircled{1}$$

$$0 + a_2 + 0 = 0 \Rightarrow a_2 = 0 \rightarrow \textcircled{2}$$

$$a_1 + 0 - a_3 = 0 \rightarrow a_1 - a_3 = 0$$

$$a_1 - a_3 = 0 \rightarrow \textcircled{3}$$

From $\textcircled{1}$ & $\textcircled{3}$

These relations hold if (only if coeff) $a_1 = a_3$

$\therefore$ Any non zero value for a_1 will do same
 $a_1 = 1$

Thus there exists scalars not all zero such that

$$a_1(1, 0, 1) + a_2(1, 1, 0) + a_3(-1, 0, -1) = (0, 0, 0)$$

i.e. for $a_1 = 1, a_2 = 0, a_3 = 1$

$$\Rightarrow 1(1, 0, 1) + 0(1, 1, 0) + 1(-1, 0, -1) = (0, 0, 0)$$

LINEAR INDEPENDENT / INDEPENDENCE

Let V be a vector space over field F . The vectors $v_1, v_2, v_3, \dots, v_n$ are said to be linearly independent over F if there exists a scalars $a_1, a_2, a_3, \dots, a_n \in F$ such that $a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n = 0$

All $a_i = 0$ when $i \in \{1, 2, 3, \dots, n\}$

example:- The vectors $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, ..., $e_n = (0, 0, \dots, 1)$ of the vector space $V_n(\mathbb{R})$ are linearly independent

88] Is possible for $a_1, a_2, a_3, \dots, a_n \in \mathbb{R}$, then

$$a_1 e_1 + a_2 e_2 + \dots + a_n e_n = 0$$

$$\Rightarrow a_1 (1, 0, 0, \dots, 0) + a_2 (0, 1, 0, \dots, 0) + \dots + a_n (0, 0, \dots, 1) = (0, 0, \dots, 0)$$

$$\Rightarrow (a_1, 0, 0, \dots, 0) + (0, a_2, \dots, 0) + \dots + (0, 0, \dots, a_n) = (0, 0, \dots, 0)$$

$$\Rightarrow (a_1, a_2, a_3, \dots, a_n) = (0, 0, 0, \dots, 0)$$

$$\Rightarrow a_1 = 0, a_2 = 0, a_3 = 0, \dots, a_n = 0$$

$\therefore$ All $a_i = 0$

$\therefore$ The given vectors are linearly independent

LINEAR SPAN.

Let S be a subset of vector space V over the field F . The set of all linear combinations of vectors in S is called a linear span of S and it is denoted by $L\{S\}$.

If $S = \emptyset$ then $L\{S\} = 0$.

PROBLEMS ON LINEAR COMBIN.

5. Write the vector $V = (1, 3, 9)$ as a linear combination of the vectors $u_1 = (2, 1, 3)$, $u_2 = (1, -1, 1)$, $u_3 = (3, 1, 5)$

88] Let $V = x u_1 + y u_2 + z u_3$

$$(1, 3, 9) = x (2, 1, 3) + y (1, -1, 1) + z (3, 1, 5)$$

$$\Rightarrow 2x + y + 3z = 1$$

$$x - y + z = 3$$

$$3x + y + 5z = 9$$

After (Applying all row trans) Rank:- No. of non zero rows of the matrix is the rank

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Now consider $AX=B$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

Augmented matrix

$$[A:B] = \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 1 & -1 & 1 & : & 3 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$[A:B] \approx \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 2 & 1 & 3 & : & 1 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$$R_2 = 2R_2 - 2R_1 \quad R_3 = R_3 - 3R_1$$

$$[A:B] \approx \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 4 & 2 & : & 0 \end{bmatrix}$$

$$R_3 = 3R_3 - 4R_2$$

$$[A:B] \approx \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 0 & 2 & : & 20 \end{bmatrix}$$

Final solution

$$\begin{aligned} x - y + z &= 3 \quad \rightarrow (1) \\ 3y + z &= -5 \quad \rightarrow (2) \\ 2z &= 20 \quad \rightarrow (3) \end{aligned}$$

$\therefore \text{Rank } A = \text{Rank } [A:B] = 3 = \text{No. of unknowns}$

$\therefore$ The system of linear eq^s is consistent & posses a unique solutions

$$x - y + z = 3 \rightarrow (1)$$

$$3y + z = -5 \rightarrow (2)$$

$$2z = 20 \rightarrow (3)$$

From (3)

$$2z = 20$$

$$z = \frac{20}{2}$$

$$z = 10$$

$$\underline{\underline{z = 10}}$$

Sub $z = 10$ in (2)

$$3y + 10 = -5$$

$$3y = -15$$

$$y = -5$$

$$\underline{\underline{y = -5}}$$

Sub y & z in (1)

$$x - (-5) + 10 = 3$$

$$x + 5 + 10 = 3$$

$$x + 15 = 3$$

$$x = 3 - 15$$

$$\underline{\underline{x = -12}}$$

$$\therefore \text{Thus } x(2, 1, 3) + y(1, -1, 1) + z(3, 1, 5) = (1, 3, 9)$$

$$-12(2, 1, 3) + (-5)(1, -1, 1) + 10(3, 1, 5) = (1, 3, 9)$$

6. Write the vectors $x = (4, 2, 1)$ as linear combinations of vectors $u_1 = (1, -3, 1)$, $u_2 = (0, 1, 2)$, $u_3 = (5, 1, 3)$

→ No linear comb)

7. Determine whether the vectors $v_1 = (1, 2, 3)$, $v_2 = (3, 1, 7)$ & $v_3 = (2, 5, 8)$ are linearly dependent or independent.

Let $xv_1 + yv_2 + zv_3 = 0$

$$x(1, 2, 3) + y(3, 1, 7) + z(2, 5, 8) = (0, 0, 0)$$

$$x + 3y + 2z = 0$$

$$2x + y + 5z = 0$$

$$3x + 7y + 8z = 0$$

Consider $AX = 0$

If unique solⁿ exists, the system eqⁿ are linearly independent. (b) The $\det(A) \neq 0$, the system of eqⁿ are linearly independent else linearly dependent. DATE

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \rightarrow \rho(A) \neq n$$

$$R_2 = R_2 - 2R_1, \quad R_3 = R_3 - 3R_1$$

$$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$R_3 = 5R_3 - 2R_2$$

$$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & 0 & 8 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \text{Rank of } A \text{ (or } \rho(A)) = 3 = 3 = \text{No. of unknowns}$

$\therefore$ Homogeneous system of linear eqⁿ possess a unique solution

$$\text{i.e. } x=0, y=0, \text{ \& } z=0$$

So vectors v_1, v_2, v_3 are linearly independent

8. Show that the vectors $\{ \overset{v_1}{(1, 1, -1)}, \overset{v_2}{(2, -3, 5)}, \overset{v_3}{(-2, 1, 4)} \}$ are linearly independent $\rightarrow$ span

Sol

Consider

$$xv_1 + yv_2 + zv_3 = 0$$

$$x(1, 1, -1) + y(2, -3, 5) + z(-2, 1, 4)$$

$$x + 2y - 2z = 0$$

$$x - 3y + 5z = 0$$

$$-x + 5y + 4z = 0$$

~~$AX = 0$~~ . Consider $AX = 0$.

~~$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 5 \\ -2 & 1 & 1 \end{bmatrix}$~~ $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -3 & 1 \\ -1 & 5 & 4 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$R_2 = R_2 - R_1 \quad R_3 = R_3 + R_1$$

$$A \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & -5 & 3 \\ 0 & 7 & 2 \end{bmatrix}$$

$$R_3 = 5R_3 + 7R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & -5 & 3 \\ 0 & 0 & 31 \end{bmatrix}$$

$\therefore f(A) = 3 = \text{No. of unknowns}$

$\Rightarrow$ i.e. $x=0, y=0, z=0$. Unique solution

$\therefore$ vectors v_1, v_2, v_3 are linearly independent

9. Determine whether the vectors $S = \{(1, 1, 2), (-3, 1, 0), (1, -1, 1), (1, 2, -3)\}$ are linearly dependent or independent

Let $xv_1 + yv_2 + zv_3 + wv_4 = 0$

$$x(1, 1, 2) + y(-3, 1, 0) + z(1, -1, 1) + w(1, 2, -3) = (0, 0, 0)$$

$$x - 3y + z + w = 0$$

$$x + y - z + 2w = 0$$

$$2x + 0 + z - 3w = 0$$

$$AX = 0$$

$$A = \begin{bmatrix} 1 & -3 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ -2 & 0 & 1 & -3 \end{bmatrix}$$

$$R_2 = R_2 - R_1, \quad R_3 = R_3 - 2R_1$$

$$A = \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & 2 & -1 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$R_2 = R_2 + 3R_1, \quad R_3 = R_3 - R_1, \quad R_4 = R_4 - R_1$$

$$A \approx \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 8 \\ 0 & -2 & -1 \\ 0 & 1 & -5 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$A \approx \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & -2 & -1 \\ 0 & 4 & 6 \end{bmatrix}$$

$$R_3 = R_3 + 2R_2, \quad R_4 = R_4 - 4R_2$$

$$A \approx \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & -11 \\ 0 & 0 & +26 \end{bmatrix}$$

$$-1 + 2(-5) =$$

$$-1 - 10 = -11$$

$$R_4 = 11R_4 + 26R_3.$$

$$A \approx \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & -11 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has 0 row hence we conclude that the given set of vectors are linearly dependent.

10. Show that the vectors $(1, 1, 2, 4)$, $(2, -1, -5, 2)$, $(1, -1, -4, 0)$ & $(2, 1, 1, 6)$ are linearly dependent in $R^4(R)$.

11. Is the set of vectors $\{(4, 0, 2, 1), (2, 1, 3, 4), (2, 3, 4, 7), (2, 3, 1, 4)\}$ linearly independent?

12. Investigate the values of λ & μ such that the set $S = \{(1, 1, 1, 6), (1, 2, 3, 10), (1, 2, \lambda, \mu)\}$ is

(a) linearly independent

(b) linearly dependent.

80 Let us consider $S = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

$$R_2 = R_2 - R_1$$

$$R_3 = R_3 - R_1$$

$$SA \approx \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$$R_3 = R_3 - R_2$$

$$SA \approx \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

(a) There are 2 conditions $\lambda \neq 3$ ensures that all 3 rows of row echelon form of matrix is non zero. Hence $\lambda \neq 3$ is the condition for the set S to be linearly independent.

(b) We will make the last row zero. Hence
 $\lambda - 3 = 0$ $\lambda - \mu - 10 = 0$
 $\therefore \mu = 30$, $\lambda = 3$ and is the condition for set to be linearly dependent.

13 ~~Show that the vectors v_1, v_2~~

(*) BASIS AND DIMENSIONS.

Basis:- Let V be a vector space over field F . The set of vectors $\{v_1, v_2, v_3, \dots, v_n\}$ is called a basis of V if

- (i) The elements $v_1, v_2, v_3, \dots, v_n$ are linearly independent
- (ii) $v_1, v_2, v_3, \dots, v_n$ span V i.e. each vector of V uniquely expressed as linear combination of $v_1, \dots, v_n$. $[L(S)] = \{c_1 v_1, c_2 v_2, \dots\}$.

Dimensions:-

The ^{basis of} no^o of elements in a vector space V is called the dimensions of V & is denoted by $\dim(V)$.

If V contains a basis with n elements, then the dimensions of $V = n$ ($\dim(V) = n$).

- (*) 13. Let W is a subspace of R^5 spanned by $x_1 = (1, 2, -1, 3, 4)$,
 $x_2 = (2, 4, -2, 6, 8)$, $x_3 = (1, 3, 2, 2, 6)$, $x_4 = (1, 4, 5, 1, 8)$
 $x_5 = (2, 7, 3, 3, 9)$. Find the subsets of vectors which form a basis of W .

$$\text{Let } W/A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1, \quad R_3 = R_3 - R_1, \quad R_4 = R_4 - R_1, \quad R_5 = R_5 - 2R_1.$$

$$W/A \approx \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$$R_4 = R_4 - 2R_3$$

$$R_5 = R_5 - 3R_3$$

$$W/A \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

 $R_2 \leftrightarrow R_3$

$$W/A \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & -4 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ No of non zero rows = 3

$\therefore \dim(A) = 3$

$\therefore \{x_1, x_2, x_3\}$ form a basis in A w.

where $x_1 = \{1, 2, -1, 3, 4\}$

$x_2 = \{0, 1, 3, -1, 2\}$

$x_3 = \{0, 0, -4, 0, -5\}$

14. Do the following set S form a basis of $\mathbb{R}^3$.

Determine the dimensions & basis of the subspace $S = \{(1, 2, 3), (3, 1, 0), (-2, 1, 3)\}$

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$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix} \quad [A] = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 3R_1 \quad = 1(3-0) - 2(9-0) + 3(3+2)$$

$$R_3 \leftarrow R_3 + 2R_1 \quad = 2(-2+3) - 3(1-2) + 1(5-0) = 0$$

$$[A] = 0$$

$\therefore S$ is linearly dependent, subset of 3 vectors of $\mathbb{R}^3$

But $\dim(\mathbb{R}^3) = 3$

$\therefore S$ is not a basis of $\mathbb{R}^3$

Q. 15. Define the term basis & dimension. Find the basis of the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$. $v_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ $v_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$
 $v_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ $v_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Also find the dimension of W

$$W = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$$

$$R_2 = R_2 + 3R_1 \quad R_3 = R_3 - 4R_1$$

$$W \approx \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & -4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix}$$

$$R_2 = \frac{1}{4} R_2 \quad R_3 = \frac{1}{5} R_3$$

$$W \approx \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & -5 & -1 & 5 \end{bmatrix}$$

$$R_3 = R_3 + R_2$$

$$W \approx \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \dim(W) = 2$ Basis $\alpha_1 = \{1 \ 6 \ 2 \ -4\}$, $\alpha_2 = \{0 \ 5 \ 1 \ -5\}$

16. Show that the vectors $(1, 1, 2, 4)$, $(2, -1, -5, 2)$, $(1, -1, -4, 0)$ and $(2, 1, 1, 6)$ are linearly dependent in $\mathbb{R}^4$ and extract a linearly independent subset. Also find the basis & dimension of the subspace by them.

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1, R_3 = R_3 - R_1, R_4 = R_4 - 2R_1$$

$$A \approx \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$$\cancel{R_3 = 3R_3 - 2R_2} \quad \cancel{R_4 = R_4 - R_2}$$

$$\cancel{A \approx \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$R_2 = \frac{1}{-3} R_2$$

$$R_3 = \frac{1}{2} R_4$$

$$A \approx \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$$R_3 = R_3 - R_2$$

$$R_4 = R_4 - R_2$$

$$A \approx \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is in row echelon form

$$\rho(A) = 2.$$

$\therefore$ The dimension of the subspace spanned by the subspace vectors is 2

$\rightarrow \mathbb{R}^2$ basis

$(1, 1, 2, 4)$ & $(2, -1, -3, 2)$ form a basis of the subspace and also these 2 form a set of linearly independent subset

Basis $= \{(1, 1, 2, 4), (2, -1, -3, 2)\}$ are the basis

Q7 Find the basis & dimension of the subspace spanned by the vectors $(2, 4, 2)$, $(1, -1, 0)$, $(1, 2, 1)$ & $(0, 3, 1)$ in $V_3(\mathbb{R})$.

Sol. Let $S = \{(2, 4, 2), (1, -1, 0), (1, 2, 1), (0, 3, 1)\}$

~~dim~~ $d(V_3(\mathbb{R})) = 3.$

$\therefore$ any subset $V_3(\mathbb{R})$ containing more than 3 vectors is linearly dependent

Now consider

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$$R_1 = \frac{1}{2} R_1$$

Linear dependence $V_3 = 2V_1 \Rightarrow \det(A) = 0$
 Standard basis of order 2 $(1,0), (0,1) \Rightarrow (2,6) \hookrightarrow$ Square matrix
 3 $(1,0,0), (0,1,0), (0,0,1) \Rightarrow (1,3,5)$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$$R_2 = R_2 - R_1 \quad R_3 = R_3 - R_1$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$R_4 = R_4 + R_2$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The final matrix has 2 non zero rows

$$\therefore \dim(S) = 2$$

and bases are $\{(1,2,1), (0,-3,-1)\}$

18. Find the basis & dimensions of the subspace spanned by the subset $S = \left\{ \begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix}, \begin{pmatrix} 2 & -4 \\ -5 & 7 \end{pmatrix}, \begin{pmatrix} 1 & -7 \\ -5 & 1 \end{pmatrix} \right\}$ of the vector space of 2×2 matrices over $\mathbb{R}$

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Let $\alpha, \beta, \gamma \in S$ all the matrices of S . Then the matrices/coordinates w.r.t standard basis are

$$(1, -5, -4, 2), (1, 1, -1, 5), (2, -4, -5, 7), (1, -7, -5, 1)$$

Now consider

$$A = \begin{bmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{bmatrix}$$

$$R_2 = R_2 - R_1 \quad R_3 = R_3 - 2R_1$$

$$R_4 = R_4 - R_1$$

$$A \approx \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{bmatrix}$$

$$R_2 = \frac{1}{3} R_2 \quad R_3 = \frac{1}{3} R_3$$

$$A \approx \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & -1 & -1 \end{bmatrix}$$

$$R_3 = R_3 - R_2 \quad R_4 = R_4 + R_2$$

$$A \approx \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2 = \dim(A)$$

$$\text{Basis of } A = \{(1, -5, -4, 2), (0, 2, 1, 1)\}$$

The basis are in matrix form $(\oplus)$

$$\begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$$

There are 2 vectors that form a basis of subspace spanned by S.

Q19 Find the basis & dimensions of the subspace spanned by the vectors $(1, 2, 0)$, $(1, 1, 1)$, $(2, 0, 1)$ of the vector space $V_3(\mathbb{Z}_3)$ where $\mathbb{Z}_3$ is the field of integer modulo 3.

89 Let $S = \{(1, 2, 0), (1, 1, 1), (2, 0, 1)\}$
 consider

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1(1) - 2(1-2) + 0(0-2) \\ &= 1 - 2(-1) \\ &= 1(\oplus)_3 2 \Rightarrow (1+2) - 3 \\ &= 0 \end{aligned}$$

$\therefore S$ is linearly dependent

Now consider $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

$$R_2 = R_2 \oplus_3 R_1$$

$$R_3 = R_3 \oplus_3 R_1$$

If multiples of 3 occurs then
we subtract with
same no

If the value is
less than modulo
then retain

DATE

$$A \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$1 \oplus_3 0 = 0$$

$$1 \oplus_3 4 = 5 - 3 = 2$$

$$1 \oplus_3 0 = 1 - 3 = -2 = 1$$

$$R_3 = R_3 \oplus_3 2R_2$$

$$0 \oplus_3 2 = 2$$

$$1 \oplus_3 0 = 1$$

$$A \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2 \oplus_3 4 = 6 - 6 = 0$$

$$1 \oplus_3 2 = 3 - 3 = 0$$

$$\rho(A) = 2 = \dim(A)$$

$$\therefore \text{Basis is } \{(1, 2, 0), (0, 2, 1)\}$$

Q6. Define

NOTE :- Basis

Any vec can be expressed as the form of standard basis

Ex:- ① standard basis of 2 are $(1, 0)$ & $(0, 1)$

$$3(1, 0) + 6(0, 1)$$

$$= (3, 6)$$

$$\cdot (3, 6)$$

② standard basis of 3 are $(1, 0, 0)$ $(0, 1, 0)$ $(0, 0, 1)$

$$= 1(1, 0, 0) + 3(0, 1, 0) + 5(0, 0, 1)$$

$$= (1, 3, 5)$$

$$\cdot (1, 3, 5)$$

LINEAR TRANSFORMATIONS.

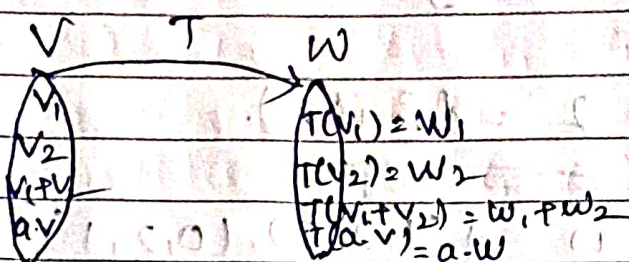
Let V & W be any 2 subspaces over the field F . A mapping $T: V \rightarrow W$ is called a linear transformation

(i) $V_1, V_2 \in V$

$$T(V_1 + V_2) = T(V_1) + T(V_2)$$

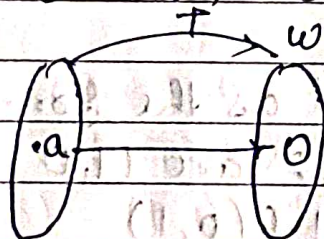
(ii) $\forall a \in F \text{ \& } V \in V$

$$T(a \cdot V) = a \cdot T(V)$$



KERNEL OF T $\text{or } \ker T$

$$\ker T = \{v \in V; T(v) = 0, 0 \in W\}$$



$$T(a) = 0$$

Any element of V which is equal to 0 then that element is called kernel

Remark

$\ker T$ is also known as NULL SPACE of T

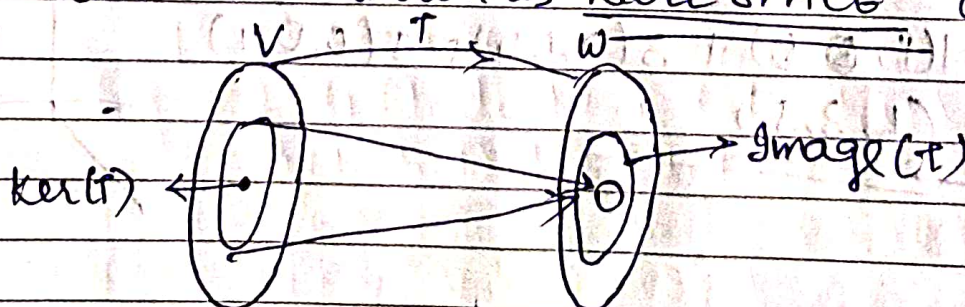


Image of T or $\text{Im}(T) = \{w \in W; w = T(v) \text{ for some } v \in V\}$

RANK NULLITY OF A LINEAR OPERATOR.

Rank of T .

Dimensions of image of T is known as Rank of T .

Nullity of T .

Dimensions of the kernel of T is known as Nullity of T .

Rank Nullity Theorem:-

Let V & W be a vector space over the field F & let T be a linear transformation from V to W if & only if V is a finite dimensional then

$$\text{rank}(T) + \text{Nullity}(T) = \text{Dimensions}(V)$$

20. For the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$. Find a basis & dimensions of its range space & null space. Also verify that $\text{rank}(T) + \text{nullity}(T) = \text{Dimensions}(\mathbb{R}^2)$.

Sol To find the nullspace of T & its dimension
By definitions of nullspace $N(T) = \{u \in \mathbb{R}^2: T(u) = 0 \in \mathbb{R}^3\}$

Let $u = (x_1, x_2) \in \mathbb{R}^2$ be an arbitrary element of nullspace
 $T(u) = 0$.

$$T(u) = T(x_1, x_2) = 0$$

$$(x_1 - x_2, x_2 - x_1, -x_1) = (0, 0, 0)$$

$$\Rightarrow x_1 - x_2 = 0$$

$$x_2 - x_1 = 0$$

$$-x_1 = 0$$

$$\Rightarrow \text{On solving } x_1 = 0 \text{ \& } x_2 = 0$$

Thus the only member of nullspace is a zero vector $\in \mathbb{R}^2$
 i.e. $N(T) = \{0\}$

$$\therefore \text{Nullity of } T = \dim\{N(T)\} = 0$$

(ii) To find the range space & its dimension
 The range space consists of ~~all~~ ordered triplets/
 triples

$(x_1 - x_2, x_2 - x_1, -x_1)$ for $(x_1, x_2) \in \mathbb{R}^2$ such
 that

$$T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1) \rightarrow \textcircled{1}$$

Let V be an element of $R(T)$.

Thus there exists $(x_1, x_2) \in \mathbb{R}^2$ such that

$$V = T(x_1, x_2)$$

$$\rightarrow (x_1, x_2) = x_1(1, 0) + x_2(0, 1) \quad (\text{standard basis})$$

$$= x_1 e_1 + x_2 e_2$$

where $e_1 = (1, 0)$ & $e_2 = (0, 1)$
 Are the standard basis of order 2.

$$T(x_1, x_2) = T(x_1 e_1 + x_2 e_2)$$

$$= x_1 T(e_1) + x_2 T(e_2) \rightarrow \textcircled{2}$$

$$T(e_1) = T(1, 0) = (1 - 0, 0 - 1, -1)$$

$$= (1, -1, -1)$$

$$T(e_2) = T(0, 1) = (0 - 1, 1 - 0, 0)$$

$$= (-1, 1, 0)$$

Eqⁿ $\textcircled{2}$ becomes

$$T(x_1, x_2) = x_1(1, -1, -1) + x_2(-1, 1, 0)$$

$$V = x_1(1, -1, -1) + x_2(-1, 1, 0) \rightarrow \textcircled{3}$$

Since $\forall v \in R(T)$ is arbitrary

(iii) Let $a(1, 1, -1) + b(-1, 1, 0) = 0$ for $a, b \in F$

$$\Rightarrow a \neq 0 \neq b \quad (a-b, -a+b, -a) = (0, 0, 0)$$

$$\Rightarrow a=b=0 \quad \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \text{ is a basis of } F^3$$

$$b-a=0 \Rightarrow b=a$$

$$-a=0 \Rightarrow a=0$$

$$a=0 \text{ \& } b=0$$

Thus $\{(1, -1, -1), (-1, 1, 0)\}$ are linearly independent

$\therefore$ it is a basis of set $R(T)$

$$\therefore \dim R(T) = 2$$

$$\text{Thus } \dim(R^2) = 1$$

$$\therefore R(T) \neq \text{Nullity}(T) \quad 2+0 = \underline{2} = \dim(R^2)$$

21. Verify the rank nullity theorem for the transformation $T: R^3 \rightarrow R^3$ defined by $T(x, y, z) = (x+2y-z, y+z, x+y-2z)$

Sol To find basis of nullity of T .

Let $v = (x, y, z) \in R^3$ such that $N(T) = \{v \in V \mid T(v) = 0\}$

$$T(x, y, z) = 0$$

$$T(x+2y-z, y+z, x+y-2z) = (0, 0, 0)$$

$$\Rightarrow \begin{cases} x+2y-z=0 \rightarrow \textcircled{1} \\ y+z=0 \rightarrow \textcircled{2} \\ x+y-2z=0 \rightarrow \textcircled{3} \end{cases} \quad \begin{array}{l} \text{from } \textcircled{2} \quad y = -z \\ x+2(-z)-z=0 \\ x-2z-z=0 \end{array}$$

On solving we get

$$x = 3z \quad y = -z$$

$$\{(x, y, z)\} = \{3z, -z, z\} = \{z(3, -1, 1)\}$$

Thus $\{(3, -1, 1)\}$ is a basis of nullity of T

$$\underline{\underline{N(T) = 1}}$$

To find the basis & range of T .

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

are the standard basis of $\mathbb{R}^3$.

$\Rightarrow \{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\}$ generates $\text{range}(T)$
 $\hookrightarrow$ substitute their values to $T(x, y, z)$

$\Rightarrow \{(1, 0, 1), (2, 1, 1), (-1, 1, -2)\}$ generates the $\text{range}(T)$

To find the basis & range

Consider a matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1, \quad R_3 = R_3 + R_1$$

$$A \approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$R_3 = R_3 - R_2$$

$$A \approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim(A) = 2$; Basis are $\{(1, 0, 1), (0, 1, -1)\}$ of $\mathbb{R}(T)$

$$\therefore \text{Rank}(T) + \text{Nullity}(T) = 2 + 1 = 3$$

$$= \dim(\mathbb{R}^3)$$

22. Verify rank nullity theorem for $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by
 $T(x, y, z) = (x+2y, y-z, x+2z)$

INNER PRODUCT SPACE.

Let V be a real vector space, suppose ^{to be} each pair of vectors $(u, v) \in V$

There is assigned a real no^r and the inner product is denoted by $\langle u, v \rangle$. This function is called a inner product on V if it satisfies the following axioms.

- (i) $\langle au + bv, v \rangle = a \langle u, v \rangle + b \langle v, v \rangle \rightarrow$ Linear property
- (ii) $\langle u, v \rangle = \langle v, u \rangle \rightarrow$ Symmetric property
- (iii) $\langle u, u \rangle \geq 0$ & $\langle u, u \rangle = 0$ iff $u = 0 \rightarrow$ Positive definite property

And it is also defined as $\mathbb{R}^n \langle u, v \rangle = u^T v$.

NORM OF A VECTOR (LENGTH OF A VECTOR).

An inner product $\langle u, u \rangle$ is non negative for any vector u
Norm of $u = \sqrt{\langle u, u \rangle} \Rightarrow \|u\| = \sqrt{\langle u, u \rangle} \Rightarrow \|u\|^2 = \langle u, u \rangle$

This ~~is also~~ non negative no^r is called norm & also called length of u

DISTANCE OF A VECTOR:-

If u & v are 2 vectors

$$d(u, v) = \|u - v\|.$$

→ Every non zero vector v in V can be multiplied by the reciprocal of its length (norm) to obtain the unit vector

$$\hat{v} = \frac{1}{\|v\|} v \text{ which is a positive multiple of } v$$

and finding of unit vector (process) is called normalising of v .

23. Define angle b/w 2 vectors from Cauchy-Schwarz inequality. The Cauchy-Schwarz inequality & Triangle inequality hold for vector $\mathbb{R}^n$. The cosine of the angle b/w 2 non zero vectors is equal to the dot product of the vectors divided by the product of their lengths.

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

NOTE: Let $[-\pi, \pi]$ be the inner product space of all continuous functions defined on closed interval $-\pi, \pi$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \cdot g(t) dt$

24. Consider the vectors $u = (1, 2, 4)$, $v = (2, -3, 5)$, $w = (4, 2, -3)$ in $\mathbb{R}^3$. Find

(i) Inner product of u, v

(ii) $\langle u, w \rangle$

(iii) $\langle v, w \rangle$

(iv) $\langle u+v, w \rangle$

(v) $\|u\|$

(vi) $\|v\|$

Sol: (i) $\langle u, v \rangle = (1)(2) + (2)(-3) + (4)(5) = 2 - 6 + 20 = 16$

(ii) $\langle u, w \rangle = (1)(4) + (2)(2) + (4)(-3) = 4 + 4 - 12 = -4$

(iii) $\langle v, w \rangle = (2)(4) + (-3)(2) + (5)(-3) = 8 - 6 - 15 = -13$

(iv) $\langle u+v, w \rangle = (3, -1, 9) \cdot (4, 2, -3) = (3)(4) + (-1)(2) + (9)(-3) = 12 - 2 - 27 = -17$

(v) $\|u\| = \sqrt{(1)^2 + (2)^2 + (4)^2} = \sqrt{21}$

$$(vi) \|v\| = \sqrt{(2)^2 + (-3)^2 + (5)^2} = \sqrt{38}$$

25. Consider the vectors $u = (1, 5)$ & $v = (3, 4)$ in $\mathbb{R}^2$ find

(i) $\langle u, v \rangle$

(ii) $\|v\|$

Sol $\langle u, v \rangle = (1)(3) + (5)(4) = 3 + 20 = 23$
 $\|v\| = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

26. Let $A = M_{2,3}$ with inner product $\langle A, B \rangle = \langle B^T, A \rangle$ & let
 $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 3 & -5 & 2 \\ 1 & 0 & -4 \end{bmatrix}$ find

(i) $\langle A, B \rangle, \langle A, C \rangle, \langle B, C \rangle$

(ii) $\langle 2A + 3B, 4C \rangle$

(iii) $\|A\|^2$ & $\|B\|$

Sol $\langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} \cdot b_{ij}$

Given $\langle A, B \rangle = \langle B^T, A \rangle$

$A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

$\langle A, B \rangle = 9 + 16 + 21 + 25 + 24 + 24 = 119$

(iv) $\langle A, C \rangle = 27 - 40 + 14 + 6 + 0 - 16 = -9$

(v) $\langle B, C \rangle = 3 - 10 + 6 + 4 + 0 - 24 = -21$

(ii) $\langle 2A + 3B, 4C \rangle =$

$$2A + 3B = \begin{bmatrix} 21 & 22 & 23 \\ 24 & 25 & 26 \end{bmatrix} \quad 4C = \begin{bmatrix} 12 & -20 & 8 \\ 4 & 0 & -16 \end{bmatrix}$$

$$\langle 2A + 3B, 4C \rangle = \begin{bmatrix} 21(12) + (22)(-20) + 23(8) + \\ 24(4) + 25(0) + 26(-16) \end{bmatrix}$$

$$= -324$$

(iii) ~~1/A~~ Now consider $\|A\|^2 = \langle A, A \rangle$

$$= \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \quad (\text{sum of squares of all elements of } A)$$

$$\|A\|^2 = \langle A, A \rangle = (9)^2 + (8)^2 + (7)^2 + (0)^2 + (5)^2 + (4)^2$$

$$= 81 + 64 + 49 + 36 + 25 + 16 = 271$$

$$\|A\| = \sqrt{271}$$

$$\|B\|^2 = \langle B, B \rangle = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$= 1 + 4 + 9 + 16 + 25 + 36$$

$$= 91$$

$$\therefore \|B\| = \sqrt{91}$$

26

Consider the following polynomials in $P(T)$ & inner product

$$f(t) = t + 2, \quad g(t) = 3t - 2, \quad h(t) = t^2 - 2t - 3 \quad \text{and} \quad \langle f, g \rangle =$$

$$\int_0^1 f(t) \cdot g(t) dt$$

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(i) Find $\langle f, g \rangle$ & $\langle g, h \rangle$.

(ii) Find $\|f\|$ & $\|g\|$

(iii) Normalise g & g .

Sol (i) $\langle f, g \rangle = \int_0^1 (t+2) \cdot (3t-2) dt$

$$= \int_0^1 (3t^2 - 2t + 6t - 4) dt$$
$$= \int_0^1 (3t^2 + 4t - 4) dt$$
$$= \left[\frac{3t^3}{3} + 4\left(\frac{t^2}{2}\right) - 4t \right]_0^1$$

$$= \frac{3(1)^3}{3} + 2(1-0) - 4(1-0)$$

$$= 1 + 2 - 4$$

$$= -1$$

$$\langle g, h \rangle = \int_0^1 (t+2) \cdot (t^2 - 2t - 3) dt$$

$$= \int_0^1 (t^3 - 2t^2 - 3t + 2t^2 - 4t - 6) dt$$

$$= \int_0^1 (t^3 - 7t - 6) dt = \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t \right]_0^1$$

$$= \frac{1}{4} (1-0) - \frac{7}{2} (1^2-0) - 6 (1-0)$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$= \frac{1-14-24}{4}$$

$$= \underline{\underline{-\frac{37}{4}}}$$

$$(ii) \|g\|^2 = \langle g, g \rangle \quad \|g\| = \sqrt{\langle g, g \rangle}$$

$$= \int_0^1 (t+2)(t+2) dt$$

$$= \int_0^1 (t^2 + 4t + 4) dt$$

$$= \left[\frac{t^3}{3} + \frac{4t^2}{2} + 4t \right]_0^1$$

$$= \frac{1(1-0)}{3} + 2(1-0) + 4(1-0)$$

$$= \frac{1}{3} + 2 + 4$$

$$= \frac{1+6+12}{3} = \underline{\underline{\frac{19}{3}}} = \|g\|^2$$

$$\text{square root} = \sqrt{\frac{19}{3}} = \underline{\underline{\frac{\sqrt{19}}{\sqrt{3}}}}$$

Normalise - to find the unit vector

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$$\|g\|^2 = \langle g, g \rangle$$

$$= \int_0^1 (3t-2)(3t-2) dt$$

$$= \frac{2(3t)(2)}{12t}$$

$$= \int_0^1 (9t^2 + 4 - 12t) dt$$

$$= \left[\frac{9}{3} t^3 + 4t - \frac{12}{2} t^2 \right]_0^1$$

$$= 3t^3 + 4t - 6t^2 \Big|_0^1$$

$$= 3(1-0) + 6(1-0) - 4(1-0)$$

$$= 3 + 6 - 4$$

$$= 5$$

$$\|g\| = \sqrt{5}$$

(iii) Since $\|g\| = \sqrt{5}$ & g is already a unit vector.

$$\hat{g} = \frac{1}{\|g\|} \cdot g = \frac{1}{\sqrt{5}} \cdot (t+2)$$

$$\hat{g} = \frac{1}{\sqrt{5}} \cdot (t+2)$$

$$\hat{g} = \frac{3}{\sqrt{5}} \cdot (t+2)$$

$$\hat{g} = \frac{1}{\|g\|} \cdot g$$

classmate

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$$= \frac{1}{1} (3t-2)$$

$$\hat{g} = \underline{\underline{3t-2}}$$

27. Find the angle b/w $\delta(t) = t-1$ & $g(t) = t$ in the polynomial space $p(t)$ with inner product $\langle \delta, g \rangle = \int_0^1 \delta(t) \cdot g(t) dt$

$$\cos \theta = \frac{\langle \delta, g \rangle}{\|\delta\| \cdot \|g\|}$$

Q8.

Find the angle b/w $(1,1)^T$ & $(1,2)^T$

Sol

Let $x = (1,1)^T$ & $y = (1,2)^T$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be 2 vectors in $\mathbb{R}^2$

$$\text{Now } \langle x, y \rangle = (1)(1) + (1)(2) = \underline{\underline{3}}$$

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$$\|x\|^2 = \langle x, x \rangle = (1)(1) + (1)(1) = 2 \implies \|x\| = \sqrt{2}$$

$$\|y\|^2 = \sqrt{\langle y, y \rangle}$$

$$\|y\|^2 = \langle y, y \rangle = 1^2 + 2^2 = \underline{\underline{5}} \implies \|y\| = \sqrt{5}$$

The angle b/w 2 vectors.

$$\cos \theta = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}$$

$$\cos \theta = \frac{3}{\sqrt{2} \cdot \sqrt{5}} = \frac{3}{\sqrt{10}}$$

$$\theta = \cos^{-1} \left(\frac{3}{\sqrt{10}} \right) = \underline{\underline{18.45^\circ}}$$

29 Suppose $u = (1, -3, 4)$ & $v = (3, 4, 7)$ find

- (i) $d(u, v)$, The distance b/w the vectors u & v
- (ii) The angle b/w the vectors u & v
- (iii) The projection of u on v

Sol Let $\vec{u} = (1, -3, 4)$ & $\vec{v} = (3, 4, 7)$

The distance b/w $\vec{u}$ & $\vec{v}$ $d(u, v) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$

$$d(\vec{u}, \vec{v}) = \sqrt{(1-3)^2 + (-3-4)^2 + (4-7)^2}$$

$$= \sqrt{(-2)^2 + (-7)^2 + (-3)^2}$$

$$= \sqrt{4 + 49 + 9} = \sqrt{62}$$

$$\therefore d(\vec{u}, \vec{v}) = \underline{\underline{\sqrt{62}}}$$

(ii)

Angle between vectors $\vec{u}$ & $\vec{v}$

Given $\vec{u} = (1, -3, 4)$ $\vec{v} = (3, 4, 7)$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

Now

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= (1)(3) + (-3)(4) + 4(7) \\ &= 3 - 12 + 28 \\ &= \underline{\underline{19}}\end{aligned}$$

$$\|\vec{u}\| = \sqrt{(1)^2 + (-3)^2 + (4)^2} = \sqrt{1 + 9 + 16} = \underline{\underline{\sqrt{26}}}$$

$$\|\vec{v}\| = \sqrt{(3)^2 + (4)^2 + (7)^2} = \sqrt{9 + 16 + 49} = \underline{\underline{\sqrt{74}}}$$

$$\cos \theta = \frac{19}{\sqrt{26} \cdot \sqrt{74}}$$

$$\cos \theta = \frac{19}{\sqrt{26} \cdot \sqrt{74}} \quad \theta = \cos^{-1} \left(\frac{19}{\sqrt{26} \cdot \sqrt{74}} \right)$$

$$\underline{\underline{\theta = 64.3^\circ}}$$

(iii)

Projection of $\vec{u}$ on $\vec{v} = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|}$

$$\underline{\underline{\frac{19}{\sqrt{74}}}}$$

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(i) w is a non empty set $w \neq \emptyset$.

(ii) $\forall x, y \in \omega; x + y \in \omega$

(iii) $\forall x \in \omega, \forall c \in \mathbb{N}, \exists c-x \in \omega$

So let us consider

(i) $(0, 0, 0) \in w$ $\because 0 = 2(0) = 3(0) = 0$. At least 0 is an element in w

(ii) $(1, 2, 3) \in \omega \quad \therefore 1 = 2(2) = 3(3) = (1, 4, 9) \in \omega \cup \omega \cup \omega$

$\therefore w$ is a subspace

31. Define vector space, write the polynomial $f(t) = at^2 + bt + c$ as a linear combination of the polynomial $p_1 = (t-1)^2$, $p_2 = (t-1)$ & $p_3 = 1$

84 The linear combination of $f(t)$ is

$$f(t) = xP_1 + yP_2 + zP_3 \longrightarrow \text{Ans. (C)}$$

$$at^2 + bt + c = x(t-1)^2 + y(t-1) + z(1)$$

$$= x(t^2 + 1 - 2t) + y\{t - 1\} + z$$

$$at^2 + bte + c = x^2 - 2x + 1 + t^2 \{x\} + t \{-2x + y\} + 1(x - y + z)$$

compare the coefficients of t^2 , t & constant

$$a = x, b = -2x + y, c = (x - y + z)$$

Sub $x = a$ in (b)

$$b = -2a + 4$$

$$y = b + 2a$$

Sub X है या नहीं

$$c = a - (b + 2a) + z$$

$$c = a - b - 2a + 2$$

$$c = -a - b + z$$

$$x = c + a + b //$$

$\therefore f(t)$ is a linear combinations of the polynomial P_1, P_2, P_3 .

eqⁿ (1) becomes

$$f(t) = a(t-1)^2 + (b+2a)(t-1) + (a+b+c)(1)$$

(*) 32. Show that the vectors $u+v, u-v$ & $u-2v+w$ are linearly independent

Sol

Consider the linear combination

$$C_1(u+v) + C_2(u-v) + C_3(u-2v+w) = 0$$

The coefficients are

$$u(C_1 + C_2 + C_3) + v(C_1 - C_2 - 2C_3) + w(C_3) = 0$$

Since u, v, w are linearly independent, the coefficients of each vector must be 0.

$$\text{The coefficients } C_1 + C_2 + C_3 = 0$$

$$C_1 - C_2 - 2C_3 = 0$$

$$C_3 = 0$$

On solving

$$C_1 + C_2 = 0$$

$$C_1 - C_2 = 0$$

$$2C_1 = 0$$

$$C_1 = 0$$

$$\Rightarrow \boxed{C_2 = 0}$$

Since the only solution is 0 coefficient

$\therefore$ The vectors $(u+v), (u-v)$ & $(u-2v+w)$ are linearly independent

MODULE-2

LINEAR ALGEBRA PART-2

Orthogonality:-

ORTHOGONAL VECTORS:- Two vectors u & v in R^n are said to be orthogonal if $\vec{u} \cdot \vec{v} = 0$ or $\vec{u}^T \vec{v} = 0 = u^T v$

Two vectors $\vec{u}$ & $\vec{v}$ are orthogonal if and only if (iff)

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 = \|u - v\|^2$$

ORTHOGONAL SETS:- A set of vectors $\{u_1, u_2, \dots, u_p\}$ in R^n is said to be an orthogonal set if each pair of distinct vector from the set is orthogonal i.e. $u_i \cdot u_j = 0$ where $i \neq j$. i.e. if $u_i - u_j = 0$ where $i \neq j$.

$$\rightarrow u_i \cdot u_j = \langle u_i, u_j \rangle = u_i^T u_j = 0$$

Example 1: show that $\{u_1, u_2, u_3\}$ is an orthogonal set where

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

Consider the 3 possible pairs of distinct vectors namely $\{u_1, u_2\}$, $\{u_1, u_3\}$ & $\{u_2, u_3\}$

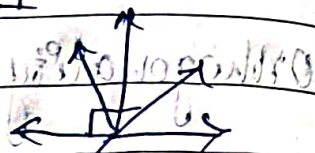
Now

$$\langle u_1, u_2 \rangle = (3)(-1) + (1)(2) + (1)(1) = -3 + 2 + 1 = 0$$

$$\begin{aligned} \langle u_1, u_3 \rangle &= (3)(-1/2) + (1)(-2) + (1)(7/2) = -3/2 - 2 + 7/2 = \\ &= \frac{-3 - 4 + 7}{2} = \frac{0}{2} = 0 \end{aligned}$$

$$\begin{aligned} \langle u_2, u_3 \rangle &= (-1)(-1/2) + (2)(-2) + (1)(7/2) = 1/2 - 4 + 7/2 = \\ &= \frac{1 - 8 + 7}{2} = \frac{0}{2} = 0 \end{aligned}$$

Each pair of distinct vectors is orthogonal & so $\{u_1, u_2, u_3\}$ is an orthogonal set. See from the figure below, the 3 line segments are mutually $\perp$



Theorem 1:-

If $S = \{u_1, \dots, u_p\}$ is an orthogonal set of non zero vectors in $\mathbb{R}^n$, then S is linearly independent & hence is a basis for the subspace spanned by S .

Proof:-

If $0 = c_1 u_1 + c_2 u_2 + \dots + c_p u_p$ for some scalar $c_1, c_2, \dots, c_p$

then

$$\begin{aligned} 0 = 0 \cdot u_1 &= (c_1 u_1 + c_2 u_2 + \dots + c_p u_p) \cdot u_1 \\ &= (c_1 u_1) \cdot u_1 + (c_2 u_2) \cdot u_1 + \dots + (c_p u_p) \cdot u_1 \\ &= c_1 \langle u_1, u_1 \rangle + c_2 \langle u_2, u_1 \rangle + \dots + c_p \langle u_p, u_1 \rangle \\ &= c_1 \langle u_1, u_1 \rangle + 0 + 0 + \dots + 0 = 0 \end{aligned}$$

($\because$ since $S = \{u_1, \dots, u_p\}$ are orthogonal set then the inner product of different vectors are 0 $\langle u_2, u_1 \rangle \dots \langle u_p, u_1 \rangle$)

$$0 = 0 \cdot u_1 = c_1 \langle u_1, u_1 \rangle$$

Because u_1 is orthogonal to $u_2, \dots, u_p$, since u_1 is non zero $\langle u_1, u_1 \rangle$ is not zero & hence so $c_1 = 0$

Similarly $c_2, c_3, \dots, c_p$ must be 0. Thus S is linearly independent.

$$y = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1$$

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ORTHOGONAL BASIS.

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

2. The set $S = \{u_1, u_2, u_3\}$ is an orthogonal set, where $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \quad \text{and } S \text{ is an orthogonal basis for } \mathbb{R}^3.$$

Express the vectors $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors in S .

Sol. We have ~~first~~ to find $y \cdot u_1$, $y \cdot u_2$ & $y \cdot u_3$.

$$y \cdot u_1 = 6(3) + 1(1) + (-8)(1) = 18 + 1 - 8 = 11$$

$$y \cdot u_2 = 6(-1) + 1(2) + (-8)(1) = -6 + 2 - 8 = -12$$

$$y \cdot u_3 = 6(-1/2) + 1(-2) + (-8)(7/2) = -3 - 2 - 28 = -33$$

$$\text{Now calculate } u_1 \cdot u_1 = (3)^2 + (1)^2 + (1)^2 = 9 + 1 + 1 = 11$$

$$u_2 \cdot u_2 = (-1)^2 + (2)^2 + (1)^2 = 1 + 4 + 1 = 6$$

$$u_3 \cdot u_3 = (-1/2)^2 + (-2)^2 + (7/2)^2 = 1/4 + 4 + 49/4 = \frac{1 + 16 + 49}{4} = \frac{66}{4} = \frac{33}{2}$$

The linear combination of y on S is

$$y = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} \cdot u_3 \quad c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1}$$

$$= \frac{11}{11} u_1 + \frac{-12}{6} u_2 + \frac{-33}{33/2} u_3$$

$$y = u_1 - u_2 - u_3$$

3. Define orthogonal basis. Show that $\{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$, then express as a linear combination of the vectors u_1, u_2, u_3 , $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

$$u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$$

$$x \cdot u_1 = (8)(1) + (-4)(0) + (-3)(1) = 8 - 0 - 3 = 5$$

$$x \cdot u_2 = (8)(-1) + (-4)(4) + (-3)(1) = -8 - 16 - 3 = -27$$

$$x \cdot u_3 = (8)(2) + (-4)(1) + (-3)(-2) = 16 - 4 + 6 = 18$$

$$u_1 \cdot u_1 = (1)^2 + (0)^2 + (1)^2 = 2$$

$$u_2 \cdot u_2 = (-1)^2 + (4)^2 + (1)^2 = 18$$

$$u_3 \cdot u_3 = (2)^2 + (1)^2 + (-2)^2 = 9$$

Then linear comb of x on s

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3} u_3$$

$$= \frac{5}{2} u_1 + \frac{-27}{18} u_2 + \frac{18}{9} u_3$$

$$= \frac{5}{2} u_1 + \left(\frac{-3}{2} \right) u_2 + 2 u_3$$

Q4. If $a = [-2, 1]^T$, $b = [-3, 1]^T$, $c = [4/3, -1, 2/3]^T$ and $d = [5, 6, -1]^T$ then compute

- (i) $\frac{(a \cdot b)}{(a \cdot a)} a$ (ii) Find a unit vector $\hat{u}$ in the direction of c . (iii) Show that d is orthogonal to c .

sol $a = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $c = \begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix}$ $d = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$

(i) $\vec{a} \cdot \vec{b} = (-2)(-3) + (1)(1) = 6 + 1 = 7$
 $\vec{a} \cdot \vec{a} = (-2)^2 + (1)^2 = 4 + 1 = 5$

$\left(\frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \right) \cdot a = \frac{7}{5} \cdot \vec{a}$
 $= \frac{7}{5} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -14/5 \\ 7/5 \end{pmatrix}$

(ii) $\hat{u} = \frac{\vec{c}}{\|\vec{c}\|}$

Now

$\|\vec{c}\| = \sqrt{(-4/3)^2 + (-1)^2 + (2/3)^2} = \sqrt{29/3}$

$\hat{u} = \frac{\begin{pmatrix} 4/3 \\ -1 \\ 2/3 \end{pmatrix}}{\sqrt{29/3}} = \begin{pmatrix} 4/\sqrt{29} \\ -3/\sqrt{29} \\ 2/\sqrt{29} \end{pmatrix}$

$\hat{u} = \frac{1}{\sqrt{29}} \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$

checking: $\left\| \begin{pmatrix} 4/\sqrt{29} \\ -3/\sqrt{29} \\ 2/\sqrt{29} \end{pmatrix} \right\| = 1$

$$(iii) \vec{d} \cdot \vec{c} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$= 5\left(\frac{4}{3}\right) + 6(-1) + (-1)\left(\frac{2}{3}\right)$$

$$= \frac{20}{3} - 6 - \frac{2}{3}$$

$$= \frac{20 - 18 - 2}{3}$$

$$= \frac{0}{3} = 0$$

$\therefore \vec{d}$ is orthogonal ($\perp$) to $\vec{c}$.

ORTHOGONAL PROJECTIONS.

Let y & u be 2 vectors in a vector space, then orthogonal projection of y on u is denoted by $\hat{y}$ and is given by

$$\hat{y} = \frac{\langle y, u \rangle}{\langle u, u \rangle} \cdot u$$

Similarly orthogonal projection of u on y is given by

$$\hat{u} = \frac{\langle u, y \rangle}{\langle y, y \rangle} \cdot y$$

and the component of y orthogonal to u is given by

$$y - \hat{y} = y - \frac{\langle y, u \rangle}{\langle u, u \rangle} \cdot u$$

Similarly component of u orthogonal to y is given by

$$u - \hat{u} = u - \frac{\langle u, y \rangle}{\langle y, y \rangle} \cdot y$$

5. Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ & $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of y on u . Then write y as the sum of 2 orthogonal vectors, one is $\text{span}\{u\}$ & one orthogonal to u . [Find the component of y orthogonal to u & also component of u orthogonal $\hat{y}$]

sol we have $\langle y, u \rangle = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$= 7(4) + 6(2) \\ = 28 + 12 \\ = 40$$

$$\langle u, u \rangle = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (4)^2 + (2)^2 = 16 + 4 = 20$$

The orthogonal projection y on u is

$$\hat{y} = \frac{\langle y, u \rangle}{\langle u, u \rangle} u$$

The component of y orthogonal to u .

$$\hat{y} = \frac{40}{20} u = 2u$$

$$\hat{y} = 2u$$

$$\hat{y} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

The sum of these 2 vectors is y

$\hat{y} + (y - \hat{y}) =$
sum of 2 vectors

$$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Verification: $\hat{y} \cdot (y - \hat{y})$ is an orthogonal

The orthogonal projection of u on y

$$\hat{u} = \frac{\langle u, y \rangle}{\langle y, y \rangle} \cdot y$$

$$\langle y, y \rangle = (-7)^2 + (6)^2 = 49 + 36 = 85$$

$$\hat{u} = \frac{40}{85} \cdot y$$

$$\hat{u} = 0.4705 \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \frac{8}{17} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 56 \\ 48 \\ 17 \end{bmatrix}$$

Component

$$u - \hat{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 56 \\ 48 \\ 17 \end{bmatrix} = \begin{bmatrix} 12 \\ -44 \\ 17 \end{bmatrix}$$

Sum of $u = \hat{u} + (u - \hat{u})$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 56 \\ 48 \\ 17 \end{bmatrix} + \begin{bmatrix} 12 \\ -44 \\ 17 \end{bmatrix}$$

6. Verify that $\{u_1, u_2\}$ is an orthogonal set. & then find the orthogonal projection of y on to $\text{span}\{u_1, u_2\}$ where
- $$u_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

Assume $\{u_1, u_2\} = W$

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Sol.

we have $\langle u_1, u_2 \rangle = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = (-1) + 1 + 0 = 0$

$\therefore \{u_1, u_2\}$ is an orthogonal set

Let $W = \text{span}\{u_1, u_2\}$. Then the orthogonal projection y on W is

$y = C_1 u_1 + C_2 u_2 \rightarrow$ ①
 $C_1 = \frac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle} \quad C_2 = \frac{\langle y, u_2 \rangle}{\langle u_2, u_2 \rangle}$

$\langle y, u_1 \rangle = (-1)(1) + (4)(1) + 3(0) = -1 + 4 + 0 = 3$

$\langle u_1, u_1 \rangle = (1)^2 + (1)^2 + (0)^2 = 1 + 1 = 2$

$\therefore C_1 = \frac{3}{2}$

$\langle y, u_2 \rangle = (-1)(-1) + (4)(1) + 3(0) = 1 + 4 + 0 = 5 \quad C_2 = 5$

$\langle u_2, u_2 \rangle = (-1)^2 + (1)^2 + (0)^2 = 1 + 1 = 2$

① becomes

$y = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix}$

$\neq (3/2 - 5/2) + (3/2 + 5/2) + (0 + 0)$

$\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$

Component of y on u

$y - \hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

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$\langle y, (y - \hat{y}) \rangle = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = (0)(0) + (0)(0) + (3)(3) = 9$

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7. $u_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$, $y = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ vector orthogonal to w

8. Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ & $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. If $\{u_1, u_2\}$ is an orthogonal basis for $w = \text{span}\{u_1, u_2\}$. Write y as the sum of vectors in w & a vector orthogonal to w .

$y = \hat{y} + (y - \hat{y})$

ORTHONORMAL SETS.

A set $\{u_1, \dots, u_p\}$ is an orthonormal set, if it is an orthogonal set of unit vectors. If w is the subspace spanned by such a set, then the set $\{u_1, \dots, u_p\}$ is an orthonormal basis for w , since the set is automatically linearly independent.

9. Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis for $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 4/\sqrt{6} \end{bmatrix}, v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

Sol. We compute $\langle v_1, v_2 \rangle = \left(\frac{3}{\sqrt{11}}\right)\left(\frac{-1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{11}}\right)\left(\frac{2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{11}}\right)\left(\frac{4}{\sqrt{6}}\right)$

$$= \frac{-3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{4}{\sqrt{66}} = \frac{-3+2+4}{\sqrt{66}} = \frac{3}{\sqrt{66}} \neq 0$$

$\langle v_1, v_3 \rangle = \left(\frac{3}{\sqrt{11}}\right)\left(\frac{-1}{\sqrt{66}}\right) + \left(\frac{1}{\sqrt{11}}\right)\left(\frac{-4}{\sqrt{66}}\right) + \left(\frac{1}{\sqrt{11}}\right)\left(\frac{7}{\sqrt{66}}\right)$

$$7. u_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \quad y = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$$

$$\langle u_1, u_2 \rangle = (-2)(0) + (3)(0) + (0)(4) = 0$$

$\therefore \{u_1, u_2\}$ is an orthogonal set

Let w be a span $\{u_1, u_2\}$, then the orthogonal projection is

$$\hat{y} = c_1 u_1 + c_2 u_2 \rightarrow \textcircled{1}$$

$$y \Rightarrow c_1 = \frac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle}$$

$$c_2 = \frac{\langle y, u_2 \rangle}{\langle u_2, u_2 \rangle}$$

$$\langle y, u_1 \rangle = (-1)(0) + (3)(3) + (6)(0) = 0 + 9 + 0 = 9$$

$$\langle u_1, u_1 \rangle = (0)^2 + (3)^2 + (0)^2 = 9$$

$$c_1 = \frac{9}{9} = 1$$

$$\langle y, u_2 \rangle = (-1)(-2) + (3)(0) + (6)(4) = 2 + 0 + 24 = 26$$

$$\langle u_2, u_2 \rangle = (-2)^2 + (0)^2 + (4)^2 = 20$$

$$c_2 = 26/20$$

$\therefore \textcircled{1}$ becomes

$$\hat{y} = 1 \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \frac{26}{20} \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -5.2/20 \\ 0 \\ 10.4/20 \end{bmatrix} = \begin{bmatrix} -13/5 \\ 3 \\ 26/5 \end{bmatrix}$$

$$y - \hat{y} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} -13/5 \\ 3 \\ 26/5 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 0 \\ 4/5 \end{bmatrix}$$

Verification: $\langle \hat{y}, (y - \hat{y}) \rangle = \begin{bmatrix} -13/5 \\ 3 \\ 26/5 \end{bmatrix} \cdot \begin{bmatrix} 8/5 \\ 0 \\ 4/5 \end{bmatrix}$

$$= (-13/5)(8/5) + (3)(0) + (26/5)(4/5)$$

$$= \left[\frac{-104}{5} + 0 + \frac{104}{5} \right] = 0$$

$$8. u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \quad u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} = y$$

$$\langle u_1, u_2 \rangle = (2)(-2) + (5)(1) + (-1)(1) = -4 + 5 - 1 = 0$$

$\therefore u_1, u_2$ are orthogonal vectors.

$$W = \text{span}\{u_1, u_2\}$$

$$y = c_1 u_1 + c_2 u_2 \rightarrow \textcircled{1}$$

$$c_1 = \frac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle} \quad c_2 = \frac{\langle y, u_2 \rangle}{\langle u_2, u_2 \rangle}$$

$$\langle y, u_1 \rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = (1)(2) + (2)(5) + (3)(-1) = 2 + 10 - 3 = 9$$

$$\langle y, u_2 \rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = (-2)(1) + (2)(1) + (3)(1) = -2 + 2 + 3 = 3$$

$$\langle u_1, u_1 \rangle = (2)^2 + (5)^2 + (-1)^2 = 4 + 25 + 1 = 30$$

$$\langle u_2, u_2 \rangle = (-2)^2 + (1)^2 + (1)^2 = 4 + 1 + 1 = 6$$

$\textcircled{1}$ becomes

$$\hat{y} = \frac{3}{10} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6/10 \\ 15/10 \\ -3/10 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2/5 \\ 7/10 \end{bmatrix}$$

$$\text{Component: } y - \hat{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2/5 \\ 7/10 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 8/5 \\ 23/10 \end{bmatrix}$$

Sum of $y = \hat{y} + (y - \hat{y})$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2/5 \\ 7/10 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 8/5 \\ 23/10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\langle v, w \rangle = 0 \rightarrow \text{Orthogonal}$$

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$$= \frac{-3}{\sqrt{726}} + \frac{-4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = 0$$

$$\langle v_2, v_3 \rangle = \left(\frac{-1}{\sqrt{6}} \right) \left(\frac{-1}{\sqrt{66}} \right) + \left(\frac{2}{\sqrt{6}} \right) \left(\frac{-4}{\sqrt{66}} \right) + \left(\frac{1}{\sqrt{6}} \right) \left(\frac{7}{\sqrt{66}} \right)$$

$$= \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0$$

$\therefore \{v_1, v_2, v_3\}$ is an orthogonal set

Also,

$$v_1 \cdot v_1 = \left(\frac{3}{\sqrt{11}} \right)^2 + \left(\frac{1}{\sqrt{11}} \right)^2 + \left(\frac{1}{\sqrt{11}} \right)^2 = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = \frac{11}{11} = 1$$

$$v_2 \cdot v_2 = \left(\frac{-1}{\sqrt{6}} \right)^2 + \left(\frac{2}{\sqrt{6}} \right)^2 + \left(\frac{1}{\sqrt{6}} \right)^2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

$$v_3 \cdot v_3 = \left(\frac{-1}{\sqrt{66}} \right)^2 + \left(\frac{-4}{\sqrt{66}} \right)^2 + \left(\frac{7}{\sqrt{66}} \right)^2 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = \frac{66}{66} = 1$$

Which shows that $\{v_1, v_2, v_3\}$ is an orthogonal set, since the set is linearly independent, its 3 vectors form a basis for $\mathbb{R}^3$.

Thus the set $\{v_1, v_2, v_3\}$ is an orthonormal set

ORTHOGONAL COMPLEMENT OF W [$W^\perp$]

Let W be any subspace of inner product space V , then the set $W^\perp = \{v \in V; \langle v, w \rangle = 0; \forall w \in W\}$ (2)

The set of all vectors in V are orthogonal to every vector in W then V is called orthogonal complement of W & denoted by $W^\perp$

10. Find a basis for $w^\perp$, where $w = \text{span}\{u_1, u_2\}$ where

$$u_1 = \begin{bmatrix} 1 \\ -5 \\ 2 \\ 3 \end{bmatrix} \quad u_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Sol

Orthogonal complement of w is $w^\perp$

$$w^\perp = \left\{ \vec{x} \text{ such that } \vec{x} \cdot \begin{bmatrix} 1 \\ -5 \\ 2 \\ 3 \end{bmatrix} = 0 \text{ \& } \vec{x} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0 \right\}$$

$$\Rightarrow x_1 - 5x_2 + 2x_3 + 3x_4 = 0 \text{ \& } 2x_1 + x_2 + x_3 + x_4 = 0$$

Coefficient matrix A

$$= \begin{bmatrix} 1 & -5 & 2 & 3 & : & 0 \\ 2 & 1 & 1 & 1 & : & 0 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1$$

$$\approx \begin{bmatrix} 1 & -5 & 2 & 3 & : & 0 \\ 0 & 11 & -3 & -5 & : & 0 \end{bmatrix}$$

$$R_2 = \frac{1}{11} R_2$$

$$\approx \begin{bmatrix} 1 & -5 & 2 & 3 & : & 0 \\ 0 & 1 & -3/11 & -5/11 & : & 0 \end{bmatrix}$$

$$2 + \left(\frac{-3}{11}\right)5$$

$$R_1 = R_1 + 5R_2$$

$$\approx \begin{bmatrix} 1 & 0 & 4/11 & 8/11 & : & 0 \\ 0 & 1 & -3/11 & -5/11 & : & 0 \end{bmatrix}$$

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identity

free variable

Now consider the free variable

$$x_3 = s \quad x_4 = t$$

1st row elements are

$$x_1 + 0x_2 + 7/11 x_3 + 8/11 x_4 = 0$$

$$\Rightarrow x_1 = -7/11 x_3 - 8/11 x_4$$

$$\Rightarrow x_1 = -\frac{7}{11} s - \frac{8}{11} t$$

from second row

$$0x_1 + x_2 - 3/11 x_3 - 5/11 x_4 = 0$$

$$\Rightarrow x_2 = \frac{3}{11} x_3 + \frac{5}{11} x_4$$

$$x_2 = \frac{3}{11} s + \frac{5}{11} t$$

Express the vector $\vec{x}$

$$\vec{x} = \begin{bmatrix} -7/11 \\ 3/11 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -8/11 \\ 5/11 \\ 0 \\ 1 \end{bmatrix} t$$

$$\text{Basis for } W^\perp = \left\{ \begin{bmatrix} -7/11 \\ 3/11 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8/11 \\ 5/11 \\ 0 \\ 1 \end{bmatrix} \right\} = \{ \vec{x} \}$$

11. Let W be the subspace of $\mathbb{R}^4$ generated by $w_1 = (1, 2, 1, 4)$ $w_2 = (1, 2, 7, 3)$. Compute $W^\perp$.

Sol Let $W = \text{span}\{w_1, w_2\} = R(A)$

$$\text{Let } A = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 2 & 7 & 3 \end{bmatrix}$$

$w^\perp$ is called the null space

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$$R_2 = R_2 - R_1$$

$$= \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 6 & -1 \end{bmatrix}$$

Leading variable $\rightarrow$ free variable

$$\text{Let } x_2 = s \quad x_4 = t$$

Now

Second row

$$0x_1 + 0x_2 + 6x_3 - x_4 = 0$$

$$6x_3 = x_4$$

$$6x_3 = t$$

$$\Rightarrow x_3 = \frac{t}{6}$$

From (1) row

$$x_1 + 2x_2 + x_3 + 4x_4 = 0$$

$$x_1 + 2s + \frac{t}{6} + 4t = 0$$

$$x_1 + 2s + \frac{25}{6}t = 0$$

$$\Rightarrow x_1 = -2s - \frac{25}{6}t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - \frac{25}{6}t \\ s \\ \frac{t}{6} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -\frac{25}{6} \\ 0 \\ \frac{1}{6} \\ 1 \end{bmatrix} t$$

$$\text{Basis for } w^\perp (\text{null space}) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{25}{6} \\ 0 \\ \frac{1}{6} \\ 1 \end{bmatrix} \right\}$$

ROTATION MATRIX. [REFLEXIVE MATRIX]

If matrix A is 2×2 matrix and if $A \cdot A^T = I$, if then 2 properties are satisfied by matrix A then A is rotation/reflexive matrix

Ex-1- Derive rotation matrix. $(0, 1) \cdot (0, 1) = (1, 0)$

Consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sqrt{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(1, 0)$

$$\Rightarrow \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2 + b^2 = 1 \quad ac + bd = 0 \Rightarrow ac = -bd$$

$$ac + bd = 0 \quad c^2 + d^2 = 1$$

squaring on both

$$a^2 c^2 = b^2 d^2$$

$$a^2(1 - d^2) = (1 - a^2)d^2$$

$$a^2 - a^2 d^2 = d^2 - a^2 d^2$$

$$a^2 = d^2 \quad \text{or}$$

$$a = \pm d$$

Now consider $ac + bd = 0$

using $d = \pm a$

$$a(c \pm b) = 0$$

Case 1 $a = 0, d = 0$

If $a = 0, a^2 + b^2 = 1$ and $d = 0$ then $c = \pm 1$

$$0 + b^2 = 1 \Rightarrow b = \pm 1$$

$$b = \pm 1$$

Possible matrix:-

$$\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$$

Case 2

$$c \pm b = 0$$

$$c = \mp b$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

$$a = \pm 1$$

$|A| = 1$ $\rightarrow$ Orthogonal matrix

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Now $a^2 + b^2 = 1$

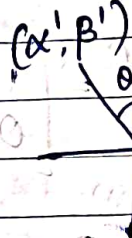
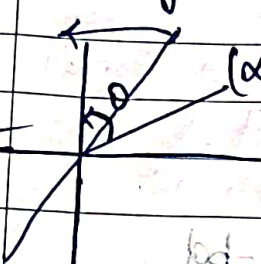
$x^2 + y^2 = 1$ represents a circle

$(x, y) = (r \cos \theta, r \sin \theta)$

$r = 1$

$(x, y) = (\cos \theta, \sin \theta)$

eigen value space



$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

$\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Rotation mat

Reflexive mat

1st matrix rotates a vector, then it is a rotation matrix
 2nd matrix reflex along the line; when we study eigen value, eigen vector, the line will give one of the eigen space

NOTE If $|A| = +1$ is a rotation matrix

If $|A| = -1$ is a reflexive matrix

* Any orthogonal matrix is rotation or reflexive matrix

⊗ EIGEN VALUES & EIGEN VECTORS.

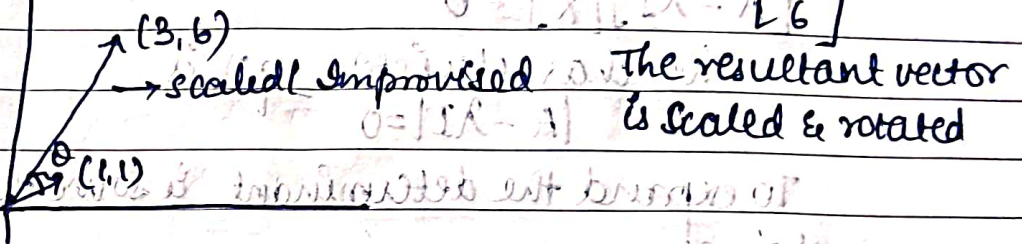
Let us take product of matrix 'A' & vector 'X' = AX .

ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

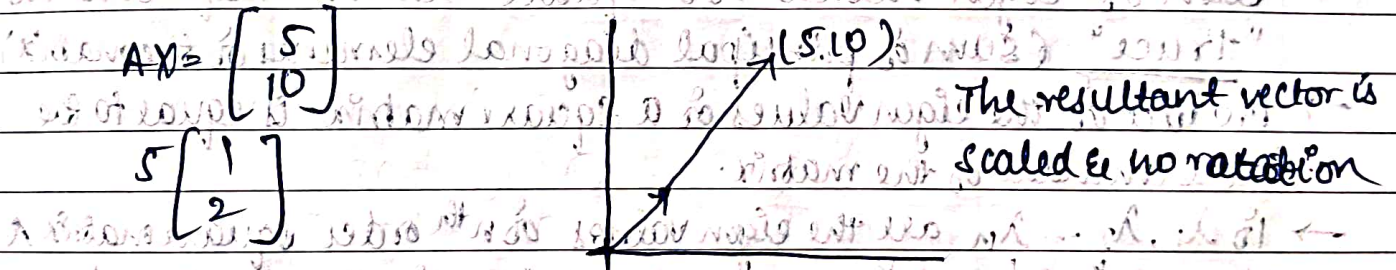
consider 2 cases

Case 1:- Let $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then $AX = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2+4 \end{bmatrix}$

Let A is a random choice not with random time $\Rightarrow \begin{bmatrix} 3 \\ 6 \end{bmatrix}$



Case 2:- Let $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then $AX = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 2+8 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$



A vector that undergoes pure scaling without any rotation is known as Eigen vector (1, 2).

The scaling factor (stretch factor) is known as eigen value (5)

Further we know that $AX = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5X$

5 is a scalar = Eigen value = stretch factor

Let us summarise, there is a vector X such that resultant X is only scaled by a factor of λ . In that case X is called eigen vector

λ is called eigen value.

Given a square matrix A , if there exists a scalar λ and a non zero column matrix X such that $AX = \lambda X$, then λ is called the eigen value of A & X is called the eigen vector of A corresponding to eigen value λ .

If I is the unit matrix of the same order as that of A , then $[A - \lambda I][X] = 0$

The characteristic eqⁿ is $|A - \lambda I| = 0$

To expand the determinant & solve the eqⁿ, we get eigen values.

PROPERTIES OF EIGEN VALUES & EIGEN VECTORS.

- Sum of eigen values of a square matrix is equal to the "trace" (sum of principal diagonal elements of the matrix)
- Product of the eigen values of a square matrix is equal to the determinant of the matrix.
- If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of n^{th} order square matrix A then $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$ are the eigen values of the matrix A^k .
- The eigen vector X of a matrix is not unique.
- If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the distinct eigen values of n^{th} order square matrix A , then the corresponding eigen vectors $X_1, X_2, \dots, X_n$ form a linearly independent set.
- If 2 or more eigen values are equal, then it may or may not be possible to get linearly independent eigen vectors corresponding to the coincident eigen values.
- If $X_1 = (x_1, y_1, z_1)$ & $X_2 = (x_2, y_2, z_2)$, then X_1, X_2 are called orthogonal vectors if $X_1 \cdot X_2 = x_1y_1 + x_2y_2 + x_3y_3 = 0$ (1)

→ The matrix $P = \left[\frac{x_1'}{\|x_1\|}, \frac{x_2'}{\|x_2\|}, \frac{x_3'}{\|x_3\|} \right]$ will be an orthogonal matrix

where $\|x\| = \sqrt{x^2 + y^2 + z^2}$

12. Find all the eigen values & corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic eqⁿ is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$(8-\lambda)[(7-\lambda)(3-\lambda)-16] + 6[-6(3-\lambda)+8] + 2[24-2(7-\lambda)] = 0$$

$$(8-\lambda)[21-7\lambda-3\lambda+\lambda^2-16] + 6[-18+6\lambda+8] + 2[24-14+2\lambda] = 0$$

$$(8-\lambda)[\lambda^2-10\lambda+5] + 6[6\lambda-10] + 2[10+2\lambda] = 0$$

$$8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 36\lambda - 60 + 20 + 4\lambda = 0$$

$$-\lambda^3 + 18\lambda^2 - 45\lambda = 0 \quad \times \ominus$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda = 0 \quad \textcircled{\text{a}} \quad \lambda^2 - 18\lambda + 45 = 0$$

$$\lambda^2 - 15\lambda - 3\lambda + 45 = 0$$

$$\lambda(\lambda - 15) - 3(\lambda - 15) = 0$$

$$\lambda = 15 \quad \textcircled{\text{b}} \quad \lambda = 3$$

Thus eigen values is $\lambda = 0, 3, 15$

We form the system of eqn from (x)

$$\begin{cases} (8-\lambda)x - 6y + 2z = 0 \\ -6x + (7-\lambda)y - 4z = 0 \\ 2x - 4y + (3-\lambda)z = 0 \end{cases} \quad \text{--- (1)}$$

Case (i) :- If $\lambda = 0$ (1) becomes

$$8x - 6y + 2z = 0 \rightarrow \text{(i)}$$

$$-6x + 7y - 4z = 0 \rightarrow \text{(ii)}$$

$$2x - 4y + 3z = 0 \rightarrow \text{(iii)}$$

Applying the rule of cross multiplication for (i) & (ii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 8 & 2 \\ -6 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{-14} = -\frac{y}{32+12} = \frac{z}{56-36}$$

$$\frac{x}{-14} = -\frac{y}{44} = \frac{z}{20} \quad \text{--- (2)}$$

$$\frac{x}{10} = -\frac{y}{20} = \frac{z}{20} \quad \text{--- (3)}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$\therefore x, y, z$ are proportional to $(1, 2, 2)$ and we can write

$$x = 1k, y = 2k, z = 2k$$

where k is arbitrary

Thus the eigen value $\lambda = 0$ corresponding eigen vector is

$$X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

case ②:- If $\lambda = 3$ (i) becomes

$$5x - 6y + 2z = 0 \rightarrow (i)$$

$$-6x + 4y - 4z = 0 \rightarrow (ii)$$

$$2x - 4y + 0z = 0 \rightarrow (iii)$$

On solving with cross multiplication for (i) & (ii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{+24+8} = \frac{-y}{-20+12} = \frac{z}{20-36}$$

$$\frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16} \times 8$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

∴ The corresponding eigenvector is

$$0 = (A - \lambda I) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$0 = (A - \lambda I) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

case ③:- If $\lambda = 15$ (i) becomes

$$-7x - 6y + 2z = 0 \rightarrow (i)$$

$$-6x - 8y - 4z = 0 \rightarrow (ii)$$

classmate $2x - 4y - 12z = 0 \rightarrow (iii)$

Applying cross rule for (P) & (ii)

$$\frac{x}{-6 \ 2} = \frac{-y}{-8 \ -4} = \frac{z}{-7 \ -6}$$

$$\frac{x}{+24+16} = \frac{-y}{28+12} = \frac{z}{+56-36}$$

$$\frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

Thus $\lambda = 15$, the corresponding eigen vector is

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

NOTE The characteristic eqn of 3rd order square matrix A can be obtained without expanding $|A - \lambda I| = 0$ by the following rules

$$\lambda^3 - (\sum d) \lambda^2 + \sum (\text{minors}) \lambda - |A| = 0$$

$\sum d$ = Sum of diagonal elements

$\sum m d$ = Sum of the minors of the diagonal elements

$$\det |A| = 0$$

For previous problem

$$\sum d = 18$$

$$\sum md = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$\begin{aligned} \text{Determinant} &= (21 - 16) + (24 - 4) + (56 - 36) \\ &= (5 + 20 + 20) \\ &= \underline{\underline{45}} \end{aligned}$$

$$\begin{aligned} |A| &= 8(-21 - 16) + 6(-18 + 8) + 2(24 - 14) \\ &= 8(5) + 6(-10) + 2(10) \\ &= 40 - 60 + 20 \end{aligned}$$

$$\underline{\underline{0}}$$

$$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

DIAGONALISATION OF A SQUARE MATRIX

If A is a square matrix of order n having n linearly independent eigen vectors, then there exists an n th order square matrix P such that $P^{-1}AP$ is a diagonal matrix.

Let A be a 3rd order square matrix having the eigen values $\lambda_1, \lambda_2, \lambda_3$ & corresponding eigen vectors $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$.

$$X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Let the modal matrix $P = [X_1, X_2, X_3]$

$$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

We calculate P^{-1} i.e.

$$P^{-1} = \frac{\text{adj}P}{|P|}$$

Finally, we get $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ = Diagonal matrix

Diagonal values are eigen values

* It is important to note that $P^{-1}AP$ is a diagonal matrix having the eigen values of A : $(\lambda_1, \lambda_2, \lambda_3)$ in its principal diagonal. We say that P diagonalises A .

NOTE ① The transformation of a square matrix A to $P^{-1}AP$ is known as similarity transformation

② The matrix P which diagonalises A is called the modal matrix of A & the resulting diagonal matrix is called spectral matrix of A .

$$A^n = PD^nP^{-1}$$

Q13. Diagonalise the matrix $\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

The characteristic eqn $|A - \lambda I| = 0$

$$\begin{vmatrix} -19-\lambda & 7 \\ -42 & 16-\lambda \end{vmatrix} = 0$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\Rightarrow [-19-\lambda](16-\lambda) + 94 + 294 = 0$$

$$\Rightarrow -304 + 19\lambda - 16\lambda + \lambda^2 + 294 = 0$$

$$\lambda^2 + 3\lambda - 10 = 0$$

$$\lambda^2 + 5\lambda - 2\lambda - 10 = 0$$

$$\lambda(\lambda+5) - 2(\lambda+5) = 0$$

$$\lambda+5=0 \quad \lambda-2=0$$

$$\lambda = -5 \quad \lambda = 2$$

$\therefore \lambda = 2, -5$ are eigen values of matrix

To find eigen vectors

Now consider $[A - \lambda I][X] = 0$

$$\begin{bmatrix} -19-\lambda & 7 \\ -42 & 16-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{cases} (-19-\lambda)x + 7y = 0 & \text{--- (P)} \\ -42x + (16-\lambda)y = 0 & \text{--- (Q)} \end{cases}$$

Put $\lambda = 2$ in (P).

$$-21x + 7y = 0$$

$$-42x + 14y = 0$$

consider $-21x + 7y = 0 \quad \div (7)$

$$-3x + y = 0$$

$$3x = y$$

$$\frac{x}{1} = \frac{y}{3}$$

$\therefore$ The eigen vectors are

$$X_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = (1, 3)'$$

$|P| \neq 0$
non singular

$|P| = 0$
singular

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Put $\lambda = -5$ in (1).

$$(-19+5)x + 7y = 0$$

$$-14x + 7y = 0$$

$$-42x + 21y = 0$$

$$-14x + 7y = 0$$

$$-42x + 21y = 0$$

consider $-14x + 7y = 0$ (2)

$$-2x + y = 0$$

$$2x = y$$

$$\frac{x}{1} = \frac{y}{2}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1, 2)'$$

$\therefore X_1, X_2$ are the eigen vectors.

The modal matrix $P = [X_1, X_2]$

$$\therefore P = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$$

To find P^{-1}

$$P^{-1} = \frac{\text{adj } P}{|P|}$$

$$|P| = 2 - 3 = -1 \neq 0$$

$$\therefore |P| = -1$$

$$(2, 1) = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\text{adj} P = \begin{bmatrix} + & - \\ - & + \end{bmatrix}' = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}'$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-1} \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\text{Now } P^{-1}AP = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -19+21 & -19+14 \\ -42+48 & -42+32 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 6 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6 & +10-10 \\ 6-6 & -15+10 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$$

14 Diagonalise the matrix $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$

$\therefore$ The characteristic eqn is $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ = 0 \end{vmatrix}$$

$$\begin{vmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (1-\lambda)(4-\lambda) + 6 &= 0 \\ -4 + \lambda - 4\lambda + \lambda^2 + 6 &= 0 \\ \lambda^2 - 3\lambda + 2 &= 0 \end{aligned}$$

$$\lambda(\lambda-2) - 1(\lambda-2) = 0$$

$$\lambda(\lambda-2) - 1(\lambda-2) = 0$$

$$\lambda - 2 = 0 \quad \lambda = 1$$

$$\lambda = 2 \quad \lambda = 1$$

$\lambda = 1, 2$ are the eigen values

Put $\lambda = 1$

$$[A - \lambda I][X] = 0$$

$$\begin{bmatrix} 1-\lambda & 3 \\ -2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{cases} (1-\lambda)x + 3y = 0 \\ -2x + (4-\lambda)y = 0 \end{cases} \rightarrow \textcircled{1}$$

Put $\lambda = 1$

$$\begin{aligned} (-1-1)x + 3y &= 0 \\ -2x + (4-1)y &= 0 \end{aligned}$$

$$\begin{aligned} -2x + 3y &= 0 \\ -2x + 3y &= 0 \end{aligned}$$

$$\text{Consider } -2x + 3y = 0$$

$$\underline{\underline{2x = 3y}}$$

$$\underline{\underline{\frac{x}{3} = \frac{y}{2}}}$$

$$x_1 = (3, 2)' = \underline{\underline{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}}$$

$$\text{Put } \lambda = 2$$

$$(4-2)x + 3y = 0$$

$$-2x + (4-2)y = 0$$

$$-3x + 3y = 0$$

$$-2x + 2y = 0$$

$$\text{Now } -3x + 3y = 0 \quad \div 3$$

$$-x + y = 0$$

$$x = y$$

$$\underline{\underline{\frac{x}{1} = \frac{y}{1}}}$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1, 1)'$$

$\therefore x_1, x_2$ are the eigenvectors.

$$\text{The modal matrix } P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\underline{\underline{|P| = 3 - 2 = 1}}$$

$$\text{adj } P = \begin{bmatrix} + & - \\ - & + \end{bmatrix}'$$

$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}'$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -3+6 & -1+3 \\ -6+8 & -2+4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & 2-2 \\ 6-6 & -4+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

= Diag(1, 2)

We can perform a normalised model matrix when all rows are equal to columns

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15. Diagonalise the matrix. Find the matrix P which transforms matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to the diagonal form.

Now $|A - \lambda I| = 0$

$$\lambda^3 - (\sum d)\lambda^2 + (\sum md)\lambda - |A| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\sum d = 1 + 5 + 1 = 7$ (Sum of diagonal)

$$\sum md = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix}$$

$$\begin{aligned} \sum md &= (5-1) + (1-9) + (5-1) \\ &= 4 + (-8) + 4 \\ &= 0 \end{aligned}$$

$$|A| = 1(5-1) - 1(1-3) + 3(1-15)$$

$$= 1(4) - 1(-2) + 3(-14)$$

$$= 4 + 2 - 42$$

$$= -36$$

$$\lambda^3 - 7\lambda^2 + 0\lambda - (-36) = 0$$

$$\lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$$

To find eigen values: By synthetic division

By inspection put $\lambda = -2$

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$(-2)^3 - 7(-2)^2 + 36 = 0$$

$$-8 - 28 + 36 = 0$$

$$0 = 0$$

Now one of the root $\lambda = -2$
The other 2 roots to be calculated by using synthetic division

$\lambda = -2$	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda - 6 - 3\lambda + 18 = 0$$

$$\lambda(\lambda - 6) - 3(\lambda - 6) = 0$$

$$\lambda - 6 = 0 \quad \lambda - 3 = 0$$

$$\lambda = 6$$

$$\lambda = 3$$

$\therefore$ The eigen values are $\lambda = -2, 3, 6$

Now consider $[A - \lambda I][X] = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 0 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$(1-\lambda)x + y + 3z = 0$$

$$x + (5-\lambda)y + 2z = 0$$

$$3x + y + (1-\lambda)z = 0$$

To find eigen vectors
Put $\lambda = -2$ in (1).

$$3x + y + 3z = 0 \quad \text{--- (1)}$$

$$x + 7y + z = 0 \quad \text{--- (2)}$$

$$3x + y + 3z = 0 \quad \text{--- (3)}$$

for (1) & (2)

On applying cross mult

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\frac{x}{1 - (21)} = \frac{-y}{3 - 3} = \frac{z}{21 - 1}$$

$$\frac{x}{-20} = \frac{-y}{0} = \frac{z}{20} \quad \times \text{ do.}$$

$$\frac{x}{-1} = \frac{-y}{0} = \frac{z}{+1}$$

$$\therefore \text{The } \underline{\underline{x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}} \text{ (1) } (-1 \ 0 \ 1) \text{ (2) } \underline{\underline{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}}$$

Put $\lambda = 3$ in (i)

$$-2x + y + 3z = 0 \rightarrow (i)$$

$$x + 2y + z = 0 \rightarrow (ii)$$

$$3x + y - 2z = 0.$$

On applying CM to (i) & (ii)

$$\begin{array}{c|c} x & z - y \\ \hline 1 & 3 \\ 2 & 1 \end{array} \quad \begin{array}{c|c} z - y & \\ \hline -2 & 3 \\ 1 & 1 \end{array} \quad \begin{array}{c|c} z - y & z \\ \hline -2 & 1 \\ 1 & 2 \end{array}$$

$$\frac{x}{(1-6)} = \frac{-y}{(-2-3)} = \frac{z}{-4-1}$$

$$\frac{x}{-5} = \frac{-y}{-5} = \frac{z}{-5}$$

$$\frac{x}{-5} = \frac{y}{5} = \frac{z}{-5}$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{-1}$$

$$x_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ or } (-1, 1, -1) \text{ or } (1, -1, 1)$$

Put $\lambda = 6$ in ①

$$-5x + y + 3z = 0$$

$$x + y + z = 0$$

$$3x + y + (-5z) = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x}{(1+3)} = \frac{-y}{(-5-3)} = \frac{z}{(5-1)}$$

$$\frac{x}{4} = \frac{-y}{-8} = \frac{z}{4} \quad \text{--- } x \text{ ②}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

∴ The model matrix is $P = [X_1 \ X_2 \ X_3]$

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

→ X_1

$\|X_1\|$

$$\|X_1\| = \sqrt{(-1)^2 + (1)^2 + 0} = \sqrt{1+1} = \sqrt{2}$$

$$P^{-1} = \frac{\text{adj } P}{|P|}$$

$$\begin{aligned}
 |P| &= -1(1+2) + 1(0-2) + 1(0-1) \\
 &= -1(3) + 1(-2) + 1(-1) \\
 &= -3 - 2 - 1 \\
 &= \underline{\underline{-6}}
 \end{aligned}$$

$$\text{adj } P = \begin{bmatrix} + (1+2) & - (0-2) & + (0-1) \\ - (-1+1) & + (-1-1) & - (1+1) \\ + (-2-1) & - (-2-0) & + (-1+0) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & -2 \\ -3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix} \quad (R_1 + R_3)$$

$$P^{-1} = \frac{1}{-6} \begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -1/3 & 1/3 & -1/3 \\ 1/6 & 1/3 & 1/6 \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -1/3 & 1/3 & -1/3 \\ 1/6 & 1/3 & 1/6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

or

we can perform normalised model matrix when we have the row elements = Column elements

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Normalised model matrix: $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

$$P^T = P^{-1} = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$P^{-1} = P^T$ because $\{x_1, x_2, x_3\}$ are orthogonal set

$$P^T A P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$0 = \lambda_2 + \lambda_3 - 3\lambda$$

$$0 = (1 + \lambda_3 - 3\lambda)\lambda$$

$$0 = (1 + \lambda_3 - 3\lambda)\lambda$$

$$0 = \lambda$$

16. Diagonalise the matrix: $A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$

The characteristic eqⁿ is $|A - \lambda I| = 0$

$$\begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{vmatrix} = 0$$

We have $\lambda^3 - (\Sigma d)\lambda^2 + (\Sigma md)\lambda - |A| = 0$
 $\Sigma d = 11 - 2 = 9 = 3$ (sum of diagonal)

$$\Sigma md = \begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -7 \\ 10 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix}$$

(minor diagonal elements)

$$= (-12 + 20) + (-66 + 70) + (-22 + 28)$$

$$= -8 + 4 + 6$$

$$= 2$$

$$|A| = 11(12 - 20) + 4(-42 + 50) - 7(-28 + 20)$$

$$= 11(-8) + 4(8) - 7(-8)$$

$$= -88 + 32 + 56$$

$$= 0$$

$\Rightarrow |A| = 0$

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

$$\lambda(\lambda^2 - 3\lambda + 2) = 0$$

$$\lambda = 0 \quad \lambda^2 - 2\lambda - \lambda + 2 = 0$$

$$\lambda(\lambda - 2) - \lambda(\lambda - 2) = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda - 2 = 0 \quad \lambda - 1 = 0$$

$$\lambda = 2 \quad \lambda = 1$$

$\therefore$ The eigen values $\lambda = 0, 1, 2$

Now consider

$$[A - \lambda I][X] = 0 \Rightarrow 0$$

$$\begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (11-\lambda)x + (-4)y - 7z = 0 \\ 7x - (2+\lambda)y - 5z = 0 \\ 10x - 4y - (6+\lambda)z = 0 \end{cases} \quad (I)$$

Put $\lambda = 0$ in (I)

$$\begin{aligned} 11x - 4y - 7z &= 0 \rightarrow (P) \\ 7x - 2y - 5z &= 0 \rightarrow (Q) \\ 10x - 4y - 6z &= 0 \rightarrow (R) \end{aligned}$$

Consider (P) & (Q) (Apply Cross Multiplication Rule)

$$\frac{x}{\begin{vmatrix} -4 & -7 \\ -2 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 11 & -7 \\ 7 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix}}$$

$$\frac{x}{20-14} = \frac{-y}{-55+49} = \frac{z}{-22+28}$$

$$\frac{x}{6} = \frac{-y}{-6} = \frac{z}{6} \quad (6)$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (1, 1, 1)'$$

Put $\lambda = 1$

$$10x - 4y - 7z = 0 \rightarrow \textcircled{i}$$

$$7x - 3y - 5z = 0 \rightarrow \textcircled{ii}$$

$$10x - 4y - 7z = 0 \rightarrow \textcircled{iii}$$

Consider $\textcircled{i}$ & $\textcircled{ii}$

$$\frac{x}{-4-7} = \frac{-y}{10-7} = \frac{z}{7-3}$$

$$\frac{x}{20-21} = \frac{-y}{-50+49} = \frac{z}{-30+28}$$

$$\frac{x}{-1} = \frac{-y}{-1} = \frac{z}{-2}$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$$

$$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ or } (1, -1, 2)'$$

Put $\lambda = 2$

$$9x - 4y - 7z = 0 \rightarrow \textcircled{i}$$

$$7x - 4y - 5z = 0 \rightarrow \textcircled{ii}$$

$$10x - 4y - 8z = 0 \rightarrow \textcircled{iii}$$

consider (i) (ii)

$$\frac{x}{\begin{vmatrix} -4 & -7 \\ -4 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 9 & -7 \\ 7 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 9 & -4 \\ 7 & -4 \end{vmatrix}}$$

$$\frac{x}{20 - 28} = \frac{-y}{-45 + 49} = \frac{z}{-36 + 28}$$

$$\frac{x}{-8} = \frac{-y}{+4} = \frac{z}{-8}$$

$$\frac{x}{-2} = \frac{y}{-1} = \frac{z}{-2}$$

$$x_3 = \begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Model matrix

$$P = [x_1 \ x_2 \ x_3] = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Minors & cofactors method

$$P^{-1} = \frac{\text{adj } P}{|P|}$$

$$|P|$$

$$\begin{aligned}
 P &= 1(-2-2) - 1(2-1) + 2(2+1) \\
 &= 1(-4) - 1(1) + 2(3) \\
 &= -4 - 1 + 6
 \end{aligned}$$

$$|P| = \underline{\underline{1}}$$

$$\text{adj } P = \begin{bmatrix} +(-4) & -1 & +3 \\ -(-2) & +0 & -1 \\ +3 & -(-1) & +(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -1 & 3 \\ 2 & 0 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj } P}{|P|} = \frac{1}{1} \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Trace:- Sum of principal diagonal elements

Model matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$

$\|X\| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

column wise

DATE

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Q. 10

Q. Diagonalise the matrix $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

Q. 11 For the matrix $A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & 4 \\ -8 & 0 & -1 \end{bmatrix}$ find $|A|$ & $\text{trace}(A)$.

$|A| = 2(-1+0) - 6(0+32) + 1(0+8)$
 $= -2 - 192 + 8$
 $= -186$

$\text{Trace}(A) = 2 + 1 - 1$ (Sum of diagonal elements)
 $= 2$

SINGULAR VALUE DECOMPOSITIONS.

A special factorisations called singular value decomposition and it is one of the most useful matrix factorisation is applied in Linear Algebra.

Let A be a $m \times n$ matrix, then square root of eigenvalues of $A^T A$ are called singular values of A and it is denoted by $\sigma_1, \sigma_2, \sigma_3, \dots$

Construction of singular value decomposition

$$\begin{bmatrix} p_1 - p_2 & p_1 + p_2 & p_1 - p_2 \\ p_1 - p_2 & p_1 + p_2 & p_1 - p_2 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is a lower triangular matrix
 $\det = (1)(1)(1) = 1$
 Eigen value

- Step 1:- Consider the given matrix A find the eigen values of $A^T A$ and corresponding orthonormal set of eigen vectors
- Step 2:- Arrange the unit eigen vectors in decreasing order on singular values i.e. $V = [X_1, X_2, X_3]$
- Step 3:- Find the square root of the eigen values called a singular values denoted by $\sigma_1, \sigma_2, \sigma_3, \dots$
- Step 4:- Form a diagonal matrix D using non zero singular values and form a matrix ' Δ ' (Σ) is same size of A with D in its upper left & with 0's else where.
- Step 5:- Construct the matrix U by using orthogonal vectors obtained from $AX_1, AX_2, \dots$ by using $u_1 = \frac{1}{\sigma_1} (AX_1), u_2 = \frac{1}{\sigma_2} (AX_2)$ and $U = [u_1, u_2]$. Number of unit vectors equal to no of rows given in the matrix.
- Step 6:- Finally the singular value decomposition is given by $A = U \Delta V^T$

Ex 19 Find the singular value decomposition of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

Sol We have $A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 16+64 & 44+56 & 56-16 \\ 44+56 & 121+49 & 154-14 \\ 56-16 & 154-14 & 196+4 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

To find eigen values of $A^T A$.

Consider $|A^T A - \lambda I| = 0$

$$\begin{vmatrix} (80-\lambda) & 100 & 40 \\ 100 & (170-\lambda) & 140 \\ 40 & 140 & (200-\lambda) \end{vmatrix} = 0$$

$$\lambda^3 - (\sum d)\lambda^2 + (\sum md)\lambda - |A| = 0$$

$$\sum d = 80 + 170 + 200 = 450$$

$$\sum md = \begin{vmatrix} 170 & 140 \\ 140 & 200 \end{vmatrix} + \begin{vmatrix} 80 & 40 \\ 40 & 200 \end{vmatrix} + \begin{vmatrix} 80 & 100 \\ 100 & 170 \end{vmatrix}$$

$$= (34000 - 19600) + (16000 - 16000) + (13600 - 10000) \\ = 32400$$

$$|A^T A| = 80(170 \cdot 200 - 140^2) - 100(20000 - 16000) + 40(14000 - 6800)$$

$$= 0$$

$$\Rightarrow \lambda^3 - 450\lambda^2 + 32400\lambda = 0$$

$$\lambda(\lambda^2 - 450\lambda + 32400) = 0$$

$$\lambda = 0 \quad \lambda^2 - 360\lambda - 90\lambda + 32400 = 0$$

$$\lambda(\lambda - 360) - 90(\lambda - 360) = 0$$

$$\lambda - 360 = 0 \quad \lambda - 90 = 0$$

$$\lambda = 360 \quad \lambda = 90$$

$\therefore$ the eigen values are $\lambda = 0, 90, 360$.

To find eigen vectors

$$[A - \lambda I][X] = 0$$

$$\begin{cases} (80 - \lambda)x + 100y + 40z = 0 \\ 100x + (170 - \lambda)y + 140z = 0 \\ 40x + 140y + (200 - \lambda)z = 0 \end{cases} \rightarrow \textcircled{1}$$

Now put $\lambda = 360$ in $\textcircled{1}$: $x(80 - 360) + 100y + 40z = 0$

$$\begin{cases} -280x + 100y + 40z = 0 \rightarrow \textcircled{p} \\ 100x - 190y + 140z = 0 \rightarrow \textcircled{ii} \\ 40x + 140y - 160z = 0 \rightarrow \textcircled{iii} \end{cases}$$

Now from $\textcircled{i}$ & $\textcircled{ii}$ $(0000 - 0000p) + (0000p - 0000s) =$

$$\frac{x}{\begin{vmatrix} 100 & 40 \\ -190 & 140 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -280 & 40 \\ 100 & 140 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -280 & 100 \\ 100 & -190 \end{vmatrix}} = \frac{1}{|A|}$$

$$\frac{x}{14000 + 7600} = \frac{-y}{-39200 - 400} = \frac{z}{53200 - 10000}$$

$$\frac{x}{21600} = \frac{-y}{43200} = \frac{z}{43200} = \frac{1}{43200}$$

$$\frac{x}{1} = \frac{-y}{2} = \frac{z}{2}$$

$$\frac{x}{1} = \frac{-y}{2} = \frac{z}{2} = \frac{1}{2}$$

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\lambda = 90$$

$$-10x + 100y + 40z = 0 \rightarrow (i)$$

$$100x + 80y + 140z = 0 \rightarrow (ii)$$

$$40x + 140y + 110z = 0 \rightarrow (iii)$$

from (i) & (ii)

$$\frac{x}{100 \ 40} = \frac{-y}{80 \ 140} = \frac{z}{-10 \ 100}$$

$$\frac{x}{10.800} = \frac{-y}{-5400} = \frac{z}{-10800}$$

$$\frac{x}{+2} = \frac{y}{+1} = \frac{z}{-2}$$

$$\therefore x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Put $\lambda = 0$

$$80x + 100y + 40z = 0 \rightarrow (i)$$

$$100x + 170y + 140z = 0 \rightarrow (ii)$$

$$40x + 140y + 200z = 0 \rightarrow (iii)$$

$$\frac{x}{100 \ 40} = \frac{-y}{80 \ 140} = \frac{z}{-10 \ 100}$$

$$\frac{x}{170 \ 140} = \frac{-y}{100 \ 140} = \frac{z}{100 \ 140}$$

$$\frac{x}{7200} = \frac{-y}{7200} = \frac{z}{3600}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

The corresponding unit eigen vectors are

$$\|X_1\| = \sqrt{(1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3$$

$$\|X_2\| = \sqrt{(2)^2 + (1)^2 + (-2)^2} = \sqrt{9} = 3$$

$$\|X_3\| = \sqrt{(2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$$

$\therefore$ The unit eigen vectors are

$$V_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$V_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \quad V_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \quad V_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \text{ are the unit eigen vectors}$$

Consider $V = [V_1, V_2, V_3]$

$$V = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

To find Δ matrix

$$\lambda_1 = 360 \quad \sigma_1 = \sqrt{360} = 6\sqrt{10}$$

$$\lambda_2 = 90 \quad \sigma_2 = \sqrt{90} = 3\sqrt{10}$$

$$\lambda_3 = 0 \quad \sigma_3 = 0 \quad \text{if } A = 0$$

$$\therefore D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$$

Δ is of same order of matrix A by using diagonal matrix

$$\Delta = [D, 0]$$

$$\Delta = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

To find U

We have $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & 2 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$$\therefore AV_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & 2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 + 22/3 + 28/3 \\ 8/3 + 14/3 - 4/3 \end{bmatrix} = \begin{bmatrix} 54/3 \\ 18/3 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

$$AV_2 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

Now consider $u_1 = \frac{AV_1}{b_1}$

$$= \frac{1}{6\sqrt{10}} \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$u_2 = \frac{AV_2}{b_2} = \frac{1}{3\sqrt{10}} \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

Hence inner product of $u_1, u_2 > 0$

$$U = [u_1 \ u_2] = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$\therefore$ Singular value decomposition of A is

$$A = U \Delta V^T$$

$$\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 10 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$O = PP^T - XP$$

$$O = PP^T - XP$$

$$O = PP^T - XP$$

$$O = PP^T - XP$$

$$P = PP^T - XP$$

$$P = PP^T - XP$$

~~Find the singular value decomposition of~~ $\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}$

Find the singular value decomposition of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

To find eigen values

$$|A^T A - \lambda I| = 0$$

$$\begin{vmatrix} 9-\lambda & -9 \\ -9 & -9-\lambda \end{vmatrix} = 0$$

$$[(9-\lambda)(9-\lambda)] - 81 = 0$$

$$9^2 - 9\lambda - 9\lambda + \lambda^2 - 81 = 0$$

$$\lambda^2 - 18\lambda = 0$$

$$\lambda(\lambda - 18) = 0$$

$$\lambda = 0 \quad \lambda = 18$$

The eigen values are $\lambda = 18$ & $\lambda = 0$

To find v

$$\begin{cases} (9-\lambda)x - 9y = 0 \\ -9x + (9-\lambda)y = 0 \end{cases} \quad (1)$$

Put $\lambda = 18$ in (1)

$$\begin{cases} -9x - 9y = 0 \rightarrow (i) \\ -9x - 9y = 0 \rightarrow (ii) \end{cases}$$

$$\Rightarrow -9x = +9y \quad (9)$$

$$\Rightarrow \frac{x}{+1} = \frac{y}{-1}$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Put $\lambda = 0$ in (1)

$$9x - 9y = 0$$

$$-9x + 9y = 0$$

Use (i)

$$9x = 9y \quad \div 9$$

$$\frac{x}{1} = \frac{y}{1}$$

$$\Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigen vectors are x_1 & x_2

Normalise x_1 & x_2 is

$$\|x_1\| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$$\|x_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\therefore v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow v = \begin{bmatrix} v_1 & v_2 \\ 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

To find Δ matrix

$$\lambda_1 = 18 \Rightarrow b_1 = \sqrt{18} = 3\sqrt{2}$$

$$\lambda_2 = 0 \Rightarrow b_2 = 0 = 0$$

$$\therefore \Delta = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} = \text{Diagonal matrix}$$

Choose Δ matrix that should have same order as A matrix

$$\Delta = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$VT = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

To find U

$$AV_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix}$$

$$u_1 = \frac{1}{b_1} AV_1$$

$$u_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

choos is not to be orthogonal to u_1 vector

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In matrix A, we have 3 rows, so we require 3 unit vectors
To find another 2 unit vectors orthogonal to u_1 - we
can use $u_1^T x = 0$

$$\begin{bmatrix} 1/3 & -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{3} - \frac{2x_2}{3} + \frac{2x_3}{3} \rightarrow \times 3$$

$$x_1 - 2x_2 + 2x_3 = 0$$

Put $x_3 = 0$

$x_2 = 1$

$\Rightarrow x_1 = 2$

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

} trial error basis

Again

Put $x_3 = 1$ $x_2 = 0 \Rightarrow x_1 = -2$

$$\therefore w_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

But both w_1 & w_2 are orthogonal to u_1

But w_1 is not orthogonal to w_2 (apply Gram Smith Process)

$\therefore$ we can find u_2 by using w_1 by normalisation

$$u_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

To find another unit vector u_3 which is orthogonal to u_1 & u_2 ^{both}

$$\text{Let } z_1 = w_2 - \frac{\langle w_1, w_2 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$z_1 = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$$

$$z_1 = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 8/5 & 4/5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 & 4/5 & 1 \end{bmatrix}$$

$$\langle w_1, w_2 \rangle =$$

$$(-2)(2) +$$

$$(0)(1) +$$

$$(1)(0) =$$

$$-4$$

$$\langle w_1, w_1 \rangle = 2^2 + 1^2 =$$

$$5$$

Normalise z_1 to get u_3 :

$$u_3 = \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

$$\|x_3\| = \sqrt{\left(\frac{-2}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + 1}$$

$$= \sqrt{\frac{45}{25}} = \frac{\sqrt{45}}{5}$$

$$u_3 = \begin{bmatrix} -2/5 / \frac{\sqrt{45}}{5} \\ 4/5 / \frac{\sqrt{45}}{5} \\ 1 / \frac{\sqrt{45}}{5} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

$$\Rightarrow \langle u_1, u_2 \rangle = 0$$

$$\langle u_2, u_3 \rangle = 0$$

$$\text{Hence } \therefore U = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 0 & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 0 & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

The singular value decomposition is

$$A = U \Delta V^T$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{3} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{3} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Q1:- Find SVD of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2+2 \\ 2+2 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

To find eigen values

$$|A^T A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)^2 - 16 = 0$$

$$25 + \lambda^2 - 10\lambda - 16 = 0$$

$$\lambda^2 - 10\lambda + 9 = 0$$

$$\lambda^2 - 9\lambda - \lambda + 9 = 0$$

$$\lambda(\lambda-9) - 1(\lambda-9) = 0$$

$$\lambda - 9 \quad \lambda - 1 = 0$$

$$\lambda = 9 \quad \lambda = 1$$

$\therefore \lambda = 9$ & $\lambda = 1$ are the eigen values
to find eigen vectors.

$$[A^T A - \lambda I] [x] = 0$$

$$\begin{cases} (5-\lambda)x - 4y = 0 \\ 4x + (5-\lambda)y = 0 \end{cases} \quad \text{--- (i)}$$

Put $\lambda = 9$ in (i)

$$-4x + 4y = 0 \rightarrow \text{--- (i)}$$

$$4x - 4y = 0 \rightarrow \text{--- (ii)}$$

$$\frac{+4x}{+4x} = \frac{4y}{4y} \quad x = y$$

$$\frac{x}{1} = \frac{y}{1}$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalise

$$\|x_1\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$V_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Put $\lambda = 1$ in (i)

$$4x + 4y = 0$$

$$4x - 4y = 0$$

$$-4x = 4y \rightarrow \text{--- (ii)}$$

$$\frac{-4x}{-4} = \frac{4y}{4}$$

$$\frac{x}{1} = \frac{y}{1}$$

classmate

$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Normalise

$$\|x_2\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$V_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = [v_1, v_2]$$

$$A^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

To find Δ

$$\Delta = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \begin{aligned} \lambda_1 &= 9 \Rightarrow \sigma_1 = \sqrt{9} = 3 \\ \lambda_2 &= 1 \Rightarrow \sigma_2 = \sqrt{1} = 1 \end{aligned}$$

$$\Delta = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

To find U

$$u_1 = \frac{A v_1}{\sigma_1}$$

$$A v_1 = u_1 = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{1} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$U = [u_1, u_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$A = U \Delta V^T$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \sqrt{3} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

1) Rotation of sample is clockwise
2) For every value of θ
3) $\theta = 0$

$$A^T A = (U \Delta V^T)^T (U \Delta V^T) = (V \Delta U^T) (U \Delta V^T) = V \Delta (U^T U) \Delta V^T = V \Delta \Delta V^T = V \Delta^2 V^T$$

MODULE-4 PROBABILITY

Exhaustive events:- An event consisting of all the various possibilities is called exhaustive events

Mutually exclusive events:- 2 or more events are said to be mutually exclusive if the happening of one event prevents the simultaneous happening of the other event

Example:- In tossing a coin, getting head & tail are mutually exclusive. In view of the fact that ^{if} head is turned out, getting tail is not possible

Independent events:- Two or more events are said to be independent if the happening or non happening of one event does not prevent the happening or non happening of the other

Probability:-

The probability of happening of event A is defined by No. of favourable events to A divided by total no. of events
Limits of probability are $0 \leq P \leq 1$

Axioms of Probability:-

If S is the sample space, & E is the set of all events, then to each event A in E , we associate a unique real no. $P = P(A)$ known as the probability of event A

- ① Probability of sample space is always 1
- ② For every event A in E , $0 \leq P(A) \leq 1$
- ③ If $A_1, A_2, A_3, \dots, A_n$ are mutually exclusive events of E , then $P(A_1 \cup A_2 \cup A_3 \dots \cup A_n) = P(A_1) + P(A_2) + P(A_3) \dots + P(A_n)$

$$P(A) = \frac{\text{No of favourable events to A}}{\text{Total no of events}} = \frac{m}{n}$$

$$P(E) = \frac{m}{n} = \frac{\text{No of favourable events to E}}{\text{Total no of events}}$$

The probability of happening is denoted by p & probability of non happening is denoted by q , then probability of ^{happening} non happening is 1

$$p + q = 1 \\ \Rightarrow q = 1 - p$$

1. The probability of getting
(i) king (ii) king or queen when a card is drawn at random from pack of 52 cards.

Sol No of possible cases (n) = 52.

- (i) No of favourable cases (m) = 4

$$\therefore \text{Probability of getting king} = \frac{m}{n} = \frac{4}{52} = \frac{1}{13}$$

- (ii) No of favourable cases (m) = 4 + 4 = 8

$$\therefore \text{Probability of getting king or queen} = \frac{m}{n} = \frac{8}{52} = \frac{2}{13}$$

2. The probability of getting

- (i) No ≥ 2 (ii) An odd no (iii) An even no when a die is thrown

Sol (i) No of favourable cases (n) = 6

(i) No of favourable outcomes $(m) = 4$ ($\because 3, 4, 5, 6$) > 2
 $\therefore$ Probability of no > 2 $(P) = \frac{4}{6} = \underline{\underline{\frac{2}{3}}}$

(ii) No of favourable outcomes $(m) = 3$ ($\because 1, 3, 5$)
 $\therefore$ Probability of odd $(P) = \frac{3}{6} = \underline{\underline{\frac{1}{2}}}$

(iii) No of favourable outcomes $(m) = 3$ ($\because 2, 4, 6$)
 $\therefore$ Probability of even $(P) = \frac{3}{6} = \underline{\underline{\frac{1}{2}}}$

ADDITION THEOREM OF PROBABILITY

The probability of happening of one or the other mutually exclusive events is equal to the sum of the probabilities of the 2 events.

That is if A, B are 2 mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

PRODUCT/MULTIPLICATION THEOREM OF PROBABILITY

If a compound event is made up of a number of independent events, the probability of happening of the compound event is equal to the product of the probabilities of the independent events.

That is if A & B are independent events, then

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

3. An Urn contains 2 white & 2 black balls and a second urn contains 2 white & 4 black balls. If one ball is drawn at random from each urn what is the probability that they are of same colour

Total no^r of balls in the first urn = 2 white + 2 Black = 4

Total no^r of balls in the second urn = 2W + 4B = 6

Case (i) Suppose both the balls drawn are white, then probability is

$$\frac{2}{4} \times \frac{2}{6} = \frac{4}{24} = \frac{1}{6}$$

Case (ii) Suppose both the balls drawn are black

$$\frac{2}{4} \times \frac{4}{6} = \frac{2}{3} = \frac{1}{3}$$

Since either of these 2 cases are favourable to the event, the required probability by addition theorem is

$$\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

Thus the required probability is $\frac{1}{2}$

4. 2 cards are drawn in succession from a pack of 52 cards. Find the probability that first king & the second is queen if the first card is

(i) Replaced

(ii) Not replaced

$$\text{8) (i) Probability of getting king} = \frac{4}{52} = \frac{1}{13}$$

$$\text{Probability of getting queen} = \frac{4}{52} = \frac{1}{13}$$

Thus

$$\text{Probability of getting king \& queen} = \frac{1}{13} \times \frac{1}{13}$$

$$= \frac{1}{169} \quad (\text{when the card is replaced})$$

(ii) In the second case as before, the probability of king is $\frac{1}{13}$

But getting queen is

If the card is not replaced, then there will be $52 - 1 = 51$ cards in the pack

and the probability of getting queen is $\frac{4}{51}$

Thus

$$\text{Probability of getting king \& queen} = \frac{1}{13} \times \frac{4}{51}$$

$$= \frac{4}{663} \quad (\text{when card is not replaced})$$

5. The probability that a person 'A' solves the problem is $\frac{1}{3}$, that of B is $\frac{1}{2}$ & that of C is $\frac{1}{5}$. If the problem is simultaneously assigned to all of them, what is the probability that the problem is solved

It should be noted that even if anyone of them solves the problem, it is presumed that the problem is solved.

We consider the following cases

$$A \text{ not solving the problem} = 1 - \frac{1}{3} = \frac{3-1}{3} = \frac{2}{3}$$

$$B \text{ not solving the problem} = 1 - \frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$$

$$C \text{ not solving the problem} = 1 - \frac{1}{5} = \frac{5-1}{5} = \frac{4}{5}$$

$$\therefore \text{The probability that the problem is not solved} = \frac{2}{3} \times \frac{1}{2} \times \frac{4}{5} = \frac{4}{15}$$

$$\text{we can } q = \frac{4}{15}$$

$$\text{But } P + Q = 1$$

$$\Rightarrow p = 1 - Q$$

$$p = 1 - \frac{4}{15}$$

$$P = \frac{15-4}{15} = \frac{11}{15}$$

6. A bag contains 10 white & 3 red balls while another bag contains 3 white & 5 red balls. 2 balls are drawn at random from the 1st bag and put it in the 2nd bag. Then a ball is drawn at random from the second bag. What is the probability that it is a white ball

SolBag B_1 :- $10W + 3R = 13$ Balls B_2 :- $3W + 5R = 8$ Balls

When 2 balls are drawn at random from the first bag,
the no of possible ways is ${}^{13}C_2$

The outcomes maybe $(W \& W)$ $(R \& R)$ $(W \& R)$

$$P(2W) = \frac{{}^{10}C_2}{{}^{13}C_2}$$

$$P(2R) = \frac{{}^3C_2}{{}^{13}C_2}$$

$$P(1W \& 1R) = \frac{10 \times 3}{{}^{13}C_2}$$

$B_2 + 2W = 5W + 5R = 10$ balls

$$P(W) = \frac{5}{10} = \frac{1}{2}$$

$B_2 + 2R = 3W + 7R = 10$ balls

$$P(R) = \frac{3}{10}$$

$B_2 + 1W + 1R = 4W + 6R = 10$ balls

$$P(W) = \frac{4}{10} = \frac{2}{5}$$

Probability in each of these cases are :-

$$\frac{{}^{10}C_2}{{}^{13}C_2} \times \frac{5}{10} ; \frac{{}^3C_2}{{}^{13}C_2} \times \frac{3}{10} ; \frac{10 \times 3}{{}^{13}C_2} \times \frac{4}{10}$$

Since either of these 3 cases are favourable to the event, the required probability is the sum of all these

Thus the probability is given by

$$= \frac{1}{{}^{13}C_2 \cdot 10} [{}^{10}C_2 \times 5 + {}^3C_2 \times 3 + 30 \times 4]$$

$$= 0.4538 \approx 0.454$$

SOME IMPORTANT LAWS IN SET OPERATIONS.

1. $A \cup B = B \cup A$; $A \cap B = B \cap A$ [Commutative]
2. $A \cup (B \cap C) = (A \cup B) \cap C$; $A \cap (B \cup C) = (A \cap B) \cup C$ [Associative]
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [Distributive]
4. $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$; $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$ [De Morgan's law]
5. $(A - B) = A \cap \bar{B}$; $\overline{(\bar{A})} = A$

Addition Rule

If A & B are any 2 events of S which are not mutually exclusive then probability of $A \cup B = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: If A & B are mutually exclusive, then probability of $A \cap B = 0$ &

$$\therefore P(A \cup B) = P(A) + P(B)$$

2. Since $P(A \cap B) \geq 0$ then $P(A \cup B) \leq P(A) + P(B)$

CONDITIONAL PROBABILITY.

If A & B are any 2 events. Probability of the happening of the event B when the event A has already happened is called

conditional probability & it is denoted by $P(B/A)$

$$P(B/A) = \frac{\text{Probability of the occurrence of both B \& A}}{\text{Probability of the occurrence of the given event A}}$$

$$\text{i.e. } P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\text{Also } P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication theorem of probability

$$P(A \cap B) = P(A) \cdot P(B|A)$$

7. (X) State & prove Bayes Theorem:
If an event A corresponds to a finite no of subevents $B_1, B_2, B_3 \dots B_n$ respectively and $P(B_i), P(A|B_i)$ are given then

$$P(B_i|A) = \frac{P(B_i) \times P(A|B_i)}{\sum_{i=1}^n P(B_i) \times P(A|B_i)}$$

Proof- Let 'S' be a sample space of a random experiment & A be an event corresponds to $B_1, B_2, B_3 \dots B_n$

Since $P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + \dots + P(A \cap B_n)$

$$\therefore P(A) = \sum_{i=1}^n P(A \cap B_i) \rightarrow (1)$$

By conditional probability

$$P(A|B_i) = \frac{P(A \cap B_i)}{P(B_i)}$$

$$\Rightarrow P(A \cap B_i) = P(B_i) \cdot P(A|B_i) \rightarrow (2)$$

Now (1) becomes

$$P(A) = \sum_{i=1}^n P(B_i) \cdot P(A|B_i) \rightarrow (3)$$

Now consider

$$P(B_i|A) = \frac{P(A \cap B_i)}{P(A)}$$

Using (2) & (3)

$$P(B_i|A) = \frac{P(B_i) \cdot P(A|B_i)}{\sum_{i=1}^n P(B_i) \cdot P(A|B_i)}$$

DATE

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8. If A & B are events with $P(A \cup B) = \frac{7}{8}$, $P(A \cap B) = \frac{1}{4}$ & $P(\bar{A}) = \frac{5}{8}$

Find $P(A)$, $P(B)$ & $P(A \cap \bar{B})$

sol we have $P(A) = 1 - P(\bar{A})$

$$= 1 - \frac{5}{8}$$

$$= \frac{8-5}{8} = \frac{3}{8}$$

$$P(B) = 1 - P(\bar{B})$$

⇒ By addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{7}{8} = \frac{3}{8} + P(B) - \frac{1}{4}$$

$$\frac{7}{8} - \frac{3}{8} + \frac{1}{4} = P(B)$$

$$P(B) = \frac{4}{8} + \frac{1}{4} = \frac{4+2}{8} = \frac{6}{8} = \frac{3}{4}$$

$$P(B) = \frac{4}{8} + \frac{1}{4} = \frac{4+2}{8} = \frac{6}{8} = \frac{3}{4}$$

$$\text{Also } P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= \frac{3}{8} - \frac{1}{4}$$

$$= \frac{1}{8}$$

9. If A & B are events with $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{4}$
 $P(\bar{B}) = \frac{5}{8}$ find the following
 $P(A \cap B)$ $P(\bar{A} \cap \bar{B})$ $P(\bar{A} \cup \bar{B})$ $P(\bar{A} \cap B)$

Sol

$$P(B) = 1 - P(\bar{B})$$

$$= 1 - \frac{5}{8}$$

$$= \frac{8-5}{8} = \frac{3}{8}$$

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{3}{4} = \frac{1}{2} + \frac{3}{8} - P(A \cap B)$$

$$\frac{3}{4} - \frac{1}{2} - \frac{3}{8} = -P(A \cap B)$$

$$\frac{6-4-3}{8} = -P(A \cap B)$$

$$\frac{6-7}{8} = -P(A \cap B)$$

$$\frac{-1}{8} = -P(A \cap B)$$

$$P(A \cap B) = \frac{1}{8}$$

(ii) $P(\bar{A} \cap \bar{B}) =$ Consider $\overline{A \cup B} = \bar{A} \cap \bar{B}$ by Demorgan's law
 $P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$

$$P(\overline{A \cup B}) = 1 - P(A \cup B)$$
$$= 1 - \frac{3}{4} = \frac{4-3}{4} = \frac{1}{4} = P(\bar{A} \cap \bar{B})$$

(iii) $P(\bar{A} \cup \bar{B})$

Now consider $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$$1 - P(A \cap B) = P(\bar{A} \cup \bar{B})$$

$$1 - \frac{1}{8} = P(\bar{A} \cup \bar{B})$$

$$\frac{7}{8} = P(\bar{A} \cup \bar{B})$$

(iv) $P(\bar{A} \cap B)$

Now $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$$= \frac{3}{8} - \frac{1}{8}$$

$$= \frac{2}{8} = \frac{1}{4}$$

Q13 In a school, 25% of the students failed in 1st Language, 15% students failed in 2nd Language & 10% of the students failed in both. If a student is selected at random, find the probability that

- (i) He failed in 1st Language if he had failed in 2nd Language
- (ii) He failed in 2nd Language if he had failed in 1st Language
- (iii) He failed in either of 2 languages

80

Let L_1 be the students failed in 1st language
 Let L_2 be the students failed in 2nd language

Given data

$$P(L_1) = \frac{25}{100} = \frac{1}{4}$$

$$P(L_2) = \frac{18}{100} = \frac{3}{20}$$

$$P(L_1 \cap L_2) = \frac{10}{100} = \frac{1}{10}$$

$$(i) P(L_1/L_2) = \frac{P(L_1 \cap L_2)}{P(L_2)} = \frac{1/10}{3/20} = \frac{2}{3}$$

$$(ii) P(L_2/L_1) = \frac{P(L_1 \cap L_2)}{P(L_1)} = \frac{1/10}{1/4} = \frac{2}{5}$$

$$(iii) P(L_1 \cup L_2) = P(L_1) + P(L_2) - P(L_1 \cap L_2)$$

$$= \frac{1}{4} + \frac{3}{20} - \frac{1}{10}$$

$$= \frac{3}{10}$$

14. A solar water heater company manufactures 2 parts namely the heating panel & the insulated tank. In the manufacturing process, 9% are likely to be defective in the panel and 5% are likely to be defective in the insulated tank. If an assembled unit is installed in a house, what is the probability that it is non defective.

8) Let $P(H)$ & $P(I)$ respectively denote the probability of heating panel & insulated tank being defective

$$\therefore \text{Given: } P(H) = \frac{9}{100} = 0.09$$

$$P(I) = \frac{5}{100} = 0.05$$

we need to find $P(\bar{H} \cap \bar{I})$ as $\bar{H}$ & $\bar{I}$ respectively stands for non heating panel & insulated tank

By De Morgan's Law: $\bar{A} \cap \bar{B} = \overline{(A \cup B)}$

$$\begin{aligned} P(\bar{H} \cap \bar{I}) &= P(\overline{H \cup I}) \\ &= 1 - P(H \cup I) \\ &= 1 - \{P(H) + P(I) - P(H \cap I)\} \\ &= 1 - \{P(H) - P(I) + P(H) \cdot P(I)\} \quad \text{By product rule} \\ &= 1 - P(H) - P(I) + \{P(H) \cdot P(I)\} \\ &= (1 - 0.09 - 0.05 + \{0.09 \cdot 0.05\}) \\ &= 0.8645 \end{aligned}$$

15. 3 machines A, B & C produce respectively 60%, 30%, 10% of the total no. of items of a factory. The percentage of defective output of these machines are respectively 2%, 3% & 4%. An item is selected at random and is found defective. Find the probability that the item was produced by the machine C.

85) Let A, B, C stand for the event of selection of an item from machines

$$\therefore P(A) = \frac{60}{100} = 0.6 \quad P(B) = \frac{30}{100} = 0.3 \quad P(C) = \frac{10}{100} = 0.1$$

Suppose D is the event of selection of an item from machine that is defective then

$$P(D/A) = \frac{2}{100} = 0.02$$

$$P(D/C) = \frac{4}{100} = 0.04$$

$$P(D/B) = \frac{3}{100} = 0.03$$

To find the probability that a selected item is produced by Machine C we need to find $P(C/D)$

By Bayes Theorem

$$P(C/D) = \frac{P(C) \cdot P(D/C)}{P(A) \cdot P(D/A) + P(B) \cdot P(D/B) + P(C) \cdot P(D/C)}$$

$$= \frac{0.1 (0.04)}{0.6(0.02) + 0.3(0.03) + 0.1(0.04)}$$

$$= 0.16$$

$$= 0.16$$

16. An office has 4 secretaries handling respectively, 20%, 60%, 15% & 5% of the files of all Govt. reports. The probability that they mis-file such reports are respectively 0.05, 0.1, 0.1 & 0.05. Find the probability that the misfiled report can be blamed on the first secretary

Let A_1, A_2, A_3 & A_4 be the 4 secretaries of the office respectively handling 20%, 60%, 15% & 5% of the office work.

Hence we have

$$P(A_1) = \frac{20}{100} = 0.2$$

$$P(A_2) = \frac{60}{100} = 0.6$$

$$P(A_3) = \frac{15}{100} = 0.15$$

$$P(A_4) = \frac{5}{100} = 0.05$$

Let E be the event of mis-filing a report by the secretary

∴ Probability

$$P(E|A_1) = 0.05$$

$$P(E|A_2) = 0.1$$

$$P(E|A_3) = 0.1$$

$$P(E|A_4) = 0.05$$

Now

To find $P(A_1|E) = \frac{P(A_1) \cdot P(E|A_1)}{P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + P(A_3) \cdot P(E|A_3) + P(A_4) \cdot P(E|A_4)}$

$$= \frac{0.2 \cdot 0.05}{0.2(0.05) + 0.6(0.1) + 0.15(0.1) + 0.05(0.05)}$$

$$= \frac{0.01}{0.01 + 0.06 + 0.015 + 0.0025}$$

$$= \frac{0.01}{0.0875} \approx 0.1142$$

The chance that a doctor will diagnose a disease correctly is 60%. The chance that a patient will die after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. If a patient dies, what is the chance that his disease was correctly diagnosed?

85] Let A be the event of correct diagnosis & B be the event of the wrong diagnosis by the doctor.

$$\therefore P(A) = \frac{60}{100} = 0.6 \quad P(B) = \frac{40}{100} = 0.4$$

Let E be the event that the patient dies

$$\therefore P(E|A) = \frac{40}{100} = 0.4 \quad P(E|B) = \frac{60}{100} = 0.6$$

We have to find $P(A|E)$

By Bayes theorem

$$P(A|E) = \frac{P(A)P(E|A)}{P(A)P(E|A) + P(B)P(E|B)}$$

$$= \frac{0.6(0.4)}{0.6(0.4) + 0.4(0.6)} \\ = 0.4615$$

18. A ball pen refills are packed in pouches containing 25 refills in each pouch. In a shop, it was found that 5 refills failed to write in pouch 1, 10 each pouches 2 & 3, one refill in pouch 4. Suppose a refill is selected at random from one of the 4 pouches. What is the probability that it fails to write? What is the probability that it was from pouch 4?

86] Let E_1, E_2, E_3 & E_4 be the event of selecting pouches at random

$$\therefore P(E_1) = \frac{1}{25} \quad P(E_2) = \frac{1}{5}$$

$$\therefore P(E_1) = \frac{1}{4} = P(E_2) = P(E_3) = P(E_4)$$

Let F be the event of selecting a refil failing to write

$$P(F|E_1) = \frac{5}{25} \quad P(F|E_2) = \frac{10}{25} \quad P(F|E_3) = \frac{10}{25}$$

$$P(F|E_4) = \frac{1}{25}$$

The probability of event F is given by

$$P(F) = P(E_1) \cdot P(F|E_1) + P(E_2) \cdot P(F|E_2) + P(E_3) \cdot P(F|E_3) + P(E_4) \cdot P(F|E_4)$$

$$= \frac{1}{4} \left[\frac{5}{25} + \frac{10}{25} + \frac{10}{25} + \frac{1}{25} \right]$$

$$= \frac{1}{4} \left[\frac{26}{25} \right]$$

$$= 0.26$$

To find $P(E_4|F)$

By Bayes Theorem

$$P(E_4|F) = \frac{P(E_4) \cdot P(F|E_4)}{P(E_1) \cdot P(F|E_1) + P(E_2) \cdot P(F|E_2) + P(E_3) \cdot P(F|E_3) + P(E_4) \cdot P(F|E_4)}$$

$$= \frac{1/4 \cdot (1/25)}{0.26}$$

$$= 0.03846$$

$$P(E_4|F) = 0.03846$$

DISCRETE AND CONTINUOUS RANDOM VARIABLE:

DISCRETE RANDOM VARIABLES:

If a random variable takes a finite or countably infinite no of values, then it is called a discrete random variable.

Ex → Tossing a coin & observing the outcome

→ Throwing a die & observing the no on the face

CONTINUOUS RANDOM VARIABLES:

If a random variable takes non countable or infinite no of values, then it is called continuous random variables.

Ex → weight of article

→ Observing the pointer on speedometer.

→ Conducting a survey on the life of electric bulbs.

Discrete Probability Distribution

If for each value ' x_i ' of a discrete random variable ' X ', we assign a real no $P(x_i)$ such that

(i) $P(x_i) \geq 0$

(ii) $\sum P(x_i) = 1$

Then the function $P(X)$ is called a probability function.

The set values $[x_i, P(x_i)]$ is called a discrete (finite) probability distribution of the discrete random variable X .

The function $P(X)$ is called probability density function (PDF) or probability mass functions (PMF).

The function $F(x)$ is defined by $F(x) = P(X \leq x) = \sum_{i=1}^x P(x_i)$
 x being the integer called cumulative distribution function (cdf).

We have the mean & variance of discrete probability distribution is defined as follows

$$\text{Mean } (\mu) = \sum_i x_i \cdot P(x_i)$$

$$\text{Variance } (V) = \sum_i (x_i - \mu)^2 \cdot P(x_i)$$

$$\text{Standard deviation } (\sigma) = \sqrt{\text{Variance}}$$

$$\sigma = \sqrt{V}$$

NOTE:- Variance can also be put in the form $\sum x_i^2 P(x_i) - \left[\sum x_i P(x_i) \right]^2$

$$V = \sum x_i^2 P(x_i) - \left[\sum x_i P(x_i) \right]^2$$

19. A coin is tossed twice. A random variable X represents the no^s of heads turning up. Find the discrete probability distribution for X . Also find its mean & variance

Sol We have $S = \{HH, HT, TH, TT\}$

The associations of elements of S to the random variable X are respectively

$$P(HH) = 1/4 \quad P(HT) = 1/4 \quad P(TH) = 1/4 \quad P(TT) = 1/4$$

$$P(X=0 \text{ i.e. no head}) = P(TT) = 1/4$$

$$P(X=1 \text{ i.e. head}) = P(HT \cup TH) = P(HT) + P(TH) \\ = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X=2 \text{ i.e. 2 head}) = P(HH) = \frac{1}{4}$$

The discrete probability distribution for X is

$X = x_i$	0	1	2
$P(x_i)$	$1/4$	$1/2$	$1/4$

We observe that $P(x_i) > 0$ & $\sum x_i = 1 = 1/4 + 1/2 + 1/4 = 1$

$$\text{Mean } (\mu) = \sum x \cdot P(x)$$

$$\begin{aligned} &= 0(1/4) + 1(1/2) + 2(1/4) \\ &= 1/2 + 1/2 \\ &= \underline{\underline{1}} \end{aligned}$$

$$\text{Variance } (V) = \sum (x_i - \mu)^2 \cdot P(x_i)$$

$$\begin{aligned} &= (0-1)^2 \left(\frac{1}{4}\right) + (1-1)^2 \left(\frac{1}{2}\right) + (2-1)^2 \left(\frac{1}{4}\right) \\ &= 1\left(\frac{1}{4}\right) + 0 + 1\left(\frac{1}{4}\right) \\ &= \underline{\underline{\frac{1}{2}}} \end{aligned}$$

$$\text{S.D} = \sqrt{V}$$

$$= \sqrt{1/2}$$

20 Show that the following distribution represents a discrete probability distribution. Find the mean & variance.

X	10	20	30	40
$P(X)$	$1/8$	$3/8$	$3/8$	$1/8$

We observe that $P(x) > 0$ & $\sum P(x) = 1$

$$\sum P(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{8}{8} = 1$$

Mean (μ) : It is a discrete probability dist

$$\begin{aligned} \text{Mean}(\mu) &= \sum x \cdot P(x) \\ &= 10\left(\frac{1}{8}\right) + 20\left(\frac{3}{8}\right) + 30\left(\frac{3}{8}\right) + 40\left(\frac{1}{8}\right) \end{aligned}$$

Variation
Standard deviation (V) = $\sum (x - \mu)^2 \cdot P(x)$

$$= (10 - 25)^2 \left(\frac{1}{8}\right) + (20 - 25)^2 \left(\frac{3}{8}\right) +$$

$$(30 - 25)^2 \left(\frac{3}{8}\right) + (40 - 25)^2 \left(\frac{1}{8}\right)$$

$$= 75$$

$$\begin{aligned} S-D &= \sqrt{V} = \sqrt{75} \\ &= 5\sqrt{3} \end{aligned}$$

Q1. Find the value of k such that the following distribution represents a finite probability distribution. Hence find its mean & S.D. Also find probability of $x \leq 1$, $P(x > 1)$ & $P(-1 < x \leq 2)$

x	-3	-2	-1	0	1	2	3
$P(x)$	k	$2k$	$3k$	$4k$	$3k$	$2k$	k

We must have $P(x) \geq 0 \quad \forall x$ and $\sum P(x) = 1$

The first condition is satisfied when $k > 0$ & the second condition is

$$k + 2k + 3k + 4k + 3k + 2k + k = 1$$

$$16k = 1$$

$$k = \frac{1}{16}$$

$$\frac{1}{16}$$

Now the discrete probability distribution

x	-3	-2	-1	0	1	2	3
$P(x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$$\text{Mean } (\mu) = \sum x_i P(x_i)$$

$$= (-3)\left(\frac{1}{16}\right) - 2\left(\frac{2}{16}\right) - 1\left(\frac{3}{16}\right) + 0\left(\frac{4}{16}\right) + 1\left(\frac{3}{16}\right) + 2\left(\frac{2}{16}\right) + 3\left(\frac{1}{16}\right)$$

$$= \frac{1}{16} (-3 - 4 - 3 + 0 + 3 + 4 + 3)$$

$$= 0$$

$$\text{Variance } (V) = \sum (x_i - \mu)^2 P(x_i)$$

$$= (-3)^2 \left(\frac{1}{16}\right) + (-2)^2 \left(\frac{2}{16}\right) + (-1)^2 \left(\frac{3}{16}\right)$$

$$+ 0 \left(\frac{4}{16}\right) + 1^2 \left(\frac{3}{16}\right) + 2^2 \left(\frac{2}{16}\right) + 3^2 \left(\frac{1}{16}\right)$$

$$= \frac{5}{2}$$

$$\text{S.D.} = \sqrt{V}$$

$$= \sqrt{\frac{5}{2}}$$

DATE

$$(i) P(X \leq 1) = P(-3) + P(-2) + P(-1) + P(0) + P(1) = 13/16$$

$$(ii) P(X > 1) = P(2) + P(3) = 2/16 + 3/16 = 3/16$$

$$(iii) P(-1 \leq X \leq 2) = P(0) + P(1) + P(2) = \frac{4}{16} + \frac{3}{16} + \frac{2}{16} = \frac{9}{16}$$

Q2. The pdf of a variety of X is given by the following data

X	0	1	2	3	4	5	6
$P(X)$	$0.1K$	$0.3K$	$0.5K$	$0.7K$	$0.9K$	$1.1K$	$1.3K$

For what value of K , this represents a valid probability distribution?

Also find $K = 1/49$

$$(i) P(X \geq 5)$$

$$(ii) P(3 < X \leq 6)$$

Q3. A random variable X has the following probability function for various values of X then

X	0	1	2	3	4	5	6	7
$P(X)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

Find

*
 Evaluate $P(X < 6)$, $P(X \geq 6)$, $P(3 < X \leq 6)$
 Also find the probability distribution functions of X .

We must have

$$P(X) \geq 0 \quad \sum P(X) = 1$$

If $k \geq 0$ & the second condⁿ

$$0 + k + 2k + 2k + 3k + \frac{k^2}{10} + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$10k^2 + 10k - k - 1 = 0$$

$$10k(k+1) - 1(k+1) = 0$$

$$10k - 1 = 0 \quad k+1 = 0$$

$$10k = 1 \quad \textcircled{2} \quad k = -1$$

$$k = 1 \quad \checkmark$$

$$\frac{10}{10}$$

If $k = -1$ the first condition fails $\therefore k = 1$

$$\frac{10}{10}$$

X	0	1	2	3	4	5	6	7
$P(X)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{1}{100}$

$$P(X < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} + \frac{2}{100}$$

$$= \frac{81}{100}$$

$$\frac{81}{100}$$

$$P(X \leq 6) = P(6) + P(7) = \frac{2}{100} + \frac{17}{100} = \frac{19}{100}$$

$$P(3 < X \leq 6) = P(4) + P(5) + P(6) = \frac{3}{10} + \frac{1}{100} + \frac{2}{100} = \frac{33}{100}$$

Probability distribution

X	0	1	2	3	4	5	6	7
$P(X)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{100}$	$\frac{81}{100}$	$\frac{83}{100}$
		$= \frac{1}{10}$	$= \frac{3}{10}$	$= \frac{5}{10}$	$= \frac{8}{10}$	$= \frac{81}{100}$	$= \frac{83}{100}$	$= \frac{100}{100}$
		$= 0.1$	$= 0.3$	$= 0.5$	$= 0.8$	$= 0.81$	$= 0.83$	$= 1$

24 A random variable X takes the values $-3, -2, -1, 0, 1, 2, 3$ such that $P(X=0) = P(X < 0)$ & $P(X=-3) = P(X=-2) = P(X=-1) = P(X=1) = P(X=2) = P(X=3)$
Find the probability distribution

80]

Let the distribution $[X, P(X)]$

X	-3	-2	-1	0	1	2	3
$P(X)$	P_1	P_2	P_3	P_4	P_5	P_6	P_7

By given data

$$P(X \geq 0) = P(X < 0)$$

$$P_4 = P_1 + P_2 + P_3 \quad \text{u.} \rightarrow \textcircled{1}$$

$$P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$$

$$P_1 = P_2 = P_3 = P_5 = P_6 = P_7 \rightarrow \textcircled{2}$$

Now

$$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 = 1 \rightarrow \textcircled{3}$$

Using $\textcircled{2}$ in $\textcircled{1}$ we get

$$P_4 = P_1 + P_1 + P_1$$

$$P_4 = 3P_1$$

$\therefore$ Using $\textcircled{1}$ in $\textcircled{3}$

$$P_1 + P_1 + P_1 + 3P_1 + P_1 + P_1 + P_1 = 1$$

$$9P_1 = 1$$

$$P_1 = \frac{1}{9}$$

$$\underline{\underline{\frac{1}{9}}}$$

$$\Rightarrow P_4 = 3P_1$$

$$P_4 = 3\left(\frac{1}{9}\right)$$

$$\underline{\underline{P_4 = \frac{1}{3}}}$$

X	-3	-2	-1	0	1	2	3
P(X)	1/9	1/9	1/9	1/3	1/9	1/9	1/9

BINOMIAL DISTRIBUTIONS

In discrete probability distributions we have 2 ~~discrete~~ distributions

→ Binomial Distribution

→ Poisson's Distributions

Binomial Distributions :- If p is the probability of success & q is the probability of failure, the probability of ' x ' success out of n trials is given by

$$P(x) = {}^n C_x p^x q^{n-x}$$

NOTE:- Binomial distributions are Bernoulli's distributions

$$\sum P(x) = q^n + nC_1 q^{n-1}p + nC_2 q^{n-2}p^2 + \dots + nC_n p^n$$

$$(q+p)^n = (1)^n = 1$$

~~Mean~~ $\mu = \sum x \cdot P(x)$

25. Derive mean & S.D of binomial distribution

Proof :- we have

$$\text{Mean}(\mu) = \sum x \cdot P(x)$$

By definition of Binomial distributions

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$= \sum x \cdot {}^n C_x p^x q^{n-x}$$

$$= \sum_{x=0}^n x \cdot \frac{n!}{(n-x)! x!} p^x q^{n-x}$$

$$= \sum_{x=0}^n x \cdot \frac{n!}{(n-x)! x(x-1)!} p^x q^{n-x}$$

$$= \sum_{x=1}^n \frac{n(n-1)!}{[(n-1)-(x-1)]!(x-1)!} p \cdot p^{x-1} \cdot q^{(n-1)-(x+1)}$$

$$\mu = np \sum_{x=1}^n \frac{(n-1)!}{[(n-1)-(x-1)]!(x-1)!} p^{x-1} q^{(n-1)-(x+1)}$$

$$\mu = np \cdot (n-1) \cdot \sum_{x=1}^n \frac{p^{x-1} q^{(n-1)-(x+1)}}{(x-1)!}$$

$$\mu = np \cdot (p+q)^{n-1}$$

$$\mu = np \cdot (1)$$

$$\underline{\underline{\mu = np}}$$

$$\text{Variance}(V) = \sum_{x=0}^n x^2 P(x) - \mu^2 \rightarrow (1)$$

Now consider

$$\sum_{x=0}^n x^2 P(x) = \sum_{x=0}^n [x(x-1) + x] P(x)$$

$$= \sum_{x=0}^n x(x-1) P(x) + \sum_{x=0}^n x \cdot P(x)$$

$$= \sum_{x=0}^n x(x-1) \frac{n!}{(n-x)! x!} p^x q^{n-x} + np \left[\sum_{x=0}^n x P(x) \right]$$

$$= \sum_{x=0}^n \frac{x(x-1) n!}{(n-x)! x!} p^x q^{n-x} + np$$

$$= \sum_{x=0}^n \frac{x(x-1) n!}{(n-x)! x(x-1)(x-2)!} p^x q^{n-x} + np$$

$$= \sum_{x=2}^n \frac{n(n-1)(n-2)!}{[(n-2)-(x-2)]!} p^2 p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^n \binom{n-2}{x-2} p^{x-2} q^{[(n-2)-(x-2)]} + np$$

$$= [n^2 - n] p^2 (p+q)^{n-2} + np$$

$$= n^2 p^2 - np^2 (1)^{n-2} + np$$

$$\sum x^2 P(x) = np^2 - np^2 + np$$

Use $\sum x^2 P(x)$ in (1)

(1) becomes

$$V = n^2 p^2 - np^2 + np - \cancel{n^2 p^2} (np)^2 \quad [\mu = np]$$

$$V = \cancel{n^2 p^2} - np^2 + np - \cancel{n^2 p^2}$$

$$V = np - np^2$$

$$V = np(1-p)$$

$$V = \underline{npq}$$

$$\text{but } p+q=1$$

$$\underline{q=1-p}$$

$$S.D = \underline{\sqrt{npq}}$$

Poisson's Distribution

The poisson's distribution of random variables is defined as

$$P(x) = \frac{m^x e^{-m}}{x!} \quad (m \rightarrow \text{Mean})$$

Mean & Standard Deviations

$$\text{Mean } (\mu) = \sum_{x=0}^{\infty} x \cdot P(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{m^x e^{-m}}{x!}$$

$$\mu = \sum_{x=0}^{\infty} x \cdot \frac{m^x e^{-m}}{x(x-1)!}$$

$$\mu = \sum_{x=1}^{\infty} \frac{m! \cdot m^{x-1} \cdot e^{-m}}{(x-1)!}$$

$$= m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= m e^{-m} \left[\frac{m^0}{0!} + \frac{m^1}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right]$$

$$= m e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right]$$

$$= m e^{-m} e^m$$

$$= m e^0$$

$$\mu = \underline{\underline{m}}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\text{Variance } (V) = \sum_{x=0}^{\infty} x^2 P(x) - \mu^2 \rightarrow (1)$$

$$\text{Now } \sum_{x=0}^{\infty} x^2 P(x) = \sum_{x=0}^{\infty} [x(x-1) + x] P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) P(x) + \sum_{x=0}^{\infty} x P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{m^x e^{-m}}{x!} + m$$

→ mean $(\mu) = m$

$$= \sum_{x=0}^{\infty} \cancel{x(x-1)} \frac{m^x e^{-m}}{\cancel{x(x-1)}(x-2)!} + m$$

$$= \sum_{x=2}^{\infty} \frac{m^2 \cdot m^{x-2} \cdot e^{-m}}{(x-2)!} + m$$

$$= m^2 e^{-m} \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$= m^2 e^{-m} \left[\frac{m^0}{0!} + \frac{m}{1!} + \frac{m^2}{2!} + \dots \right] + m$$

$$= m^2 e^{-m} \left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right] + m$$

$$= m^2 e^{-m} \cdot e^m + m = m^2 + m$$

(i) becomes

$$\text{Variance (V)} = \sum_{x=0}^{\infty} x^2 P(x) - \mu^2$$

$$= m^2 + m - m^2 = m$$

$$= m$$

$$\text{Mean } (\mu) = m = \text{Variance}$$

$$\text{Standard Deviation} = \sqrt{V} = \sqrt{m}$$

Q5. When a coin is tossed n times, find the probability of getting

- (i) exactly 1 head
- (ii) At most 3 heads
- (iii) At least 2 heads

$$p = P(H) = \frac{1}{2} = 0.5$$

$$q = P(T) = \frac{1}{2} = 0.5$$

$$n = 6$$

classmate

we have probability of x success out of n trials is given by

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$(i) P(\text{1 head}) = P(x=1) = {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{4-1}$$

$$= {}^4C_1 \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^3$$

$$= 0.25$$

$$\begin{aligned} (ii) P(\text{at most 3 heads}) &= P(x=0) + P(x=1) + P(x=2) + P(x=3) \\ &= {}^4C_0 (0.5)^0 (0.5)^4 + {}^4C_1 (0.5)^1 (0.5)^3 + {}^4C_2 (0.5)^2 (0.5)^2 + {}^4C_3 (0.5)^3 (0.5)^1 \\ &= (0.5)^4 (1 + 4 + 6 + 4) \\ &= \underline{\underline{0.9375}} \end{aligned}$$

$$\begin{aligned} (iii) P(\text{at least 2 heads}) &= P(2) + P(3) + P(4) + P(5) \text{ or} \\ &= 1 - [P(x=0) + P(x=1)] \\ &= 1 - [(0.5)^4 + {}^4C_1 (0.5)(0.5)^3] \\ &= \underline{\underline{0.6875}} \end{aligned}$$

we have probability of x success out of n trials is given by

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$(i) P(\pm \text{ head}) = P(x=1) = {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{4-1}$$

$$= {}^4C_1 \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^3$$

$$= 0.25$$

$$\begin{aligned} (ii) P(\text{at most 3 heads}) &= P(x=0) + P(x=1) + P(x=2) + P(x=3) \\ &= {}^4C_0 (0.5)^0 (0.5)^4 + {}^4C_1 (0.5) (0.5)^3 + {}^4C_2 (0.5)^2 (0.5)^2 + {}^4C_3 (0.5) (0.5)^3 \\ &= (0.5)^4 (1 + 4 + 6 + 4) \\ &= \underline{\underline{0.9375}} \end{aligned}$$

$$\begin{aligned} (iii) P(\text{at least 2 heads}) &= P(x=2) + P(x=3) + P(x=4) \text{ or} \\ &= 1 - [P(x=0) + P(x=1)] \\ &= 1 - [(0.5)^4 + {}^4C_1 (0.5) (0.5)^3] \\ &= \underline{\underline{0.6875}} \end{aligned}$$

CONTINUOUS PROBABILITY DISTRIBUTION

For every x belonging to random variable X we assign a real number $f(x)$ satisfying the condition

$$(i) f(x) \geq 0$$

$$(ii) \int_{-\infty}^{\infty} f(x) dx = 1$$

Then $f(x)$ is called a continuous probability function (p.d.f.)

Mean & Variance:-

If X is a continuous random variable with probability density function $f(x)$ where x lies b/w $-\infty$ to $+\infty$

The mean (or) expectation $[E(X)]$ and the variance (σ^2) of X is defined as follows

$$\text{Mean } (\mu) = E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$\text{variance } (\sigma^2) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

In continuous probability distribution, there are 2 distributions

→ Exponential distribution

→ Normal distributions

EXPONENTIAL DISTRIBUTION

The continuous prob distribution having the p.d.f $f(x)$ is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{is an exponential distribution}$$

Find the mean & variance of an exponential distribution

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$= \int_0^{\infty} x \cdot \alpha e^{-\alpha x} dx$$

$$\because f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \alpha \int_0^{\infty} x e^{-\alpha x} dx$$

$\therefore$ Applying Bernoulli's rule

$$\int u v = u v_1 - u' v_2 + u'' v_3 + \dots$$

$$\mu = \alpha \left[\frac{x e^{-\alpha x}}{-\alpha} - (1) \left(\frac{-1}{\alpha} \right) \frac{e^{-\alpha x}}{-\alpha} \right]_0^{\infty}$$

$$v_1 = \int e^{-\alpha x} dx = \frac{e^{-\alpha x}}{-\alpha}$$

$$\mu = \alpha \left[\frac{1}{\alpha} \{0-0\} - \frac{1}{\alpha^2} \{0-1\} \right]$$

$$\mu = \left[\frac{\alpha}{\alpha^2} \right]$$

$$\mu = \frac{1}{\alpha}$$

$$\text{variance}(\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

$$\sigma^2 = \int_0^{\infty} (x-\mu)^2 \alpha e^{-\alpha x} dx \rightarrow \text{By defn}$$

$$\sigma^2 = \alpha \int_0^{\infty} (x-\mu)^2 e^{-\alpha x} dx$$

Apply Bernoulli's rule

$$\sigma^2 = \alpha \left[(x-\mu)^2 \frac{e^{-\alpha x}}{-\alpha} - 2(x-\mu)(1) \left(\frac{-1}{\alpha} \right) \left(\frac{e^{-\alpha x}}{-\alpha} \right) + 2 \left(\frac{1}{\alpha^2} \right) \left(\frac{e^{-\alpha x}}{-\alpha} \right) \right]_0^{\infty}$$

$$\sigma^2 = \alpha \left[\{0 - (0-\mu)^2\} \frac{e^{-0}}{-\alpha} + 2 \{0 - (0-\mu) e^{-0}\} + \frac{2}{\alpha^3} \{0 - 1\} \right]$$

$$\sigma^2 = \alpha \left[\frac{\mu^2}{\alpha} (1) + \frac{2\mu}{\alpha^2} + \frac{2}{\alpha^3} \right]$$

$$\text{but } \mu = \frac{1}{\alpha}$$

$$\sigma^2 = \cancel{\alpha} \left[\frac{1/\alpha^2}{\cancel{\alpha}} - \frac{2}{\cancel{\alpha^2}} \cdot \frac{1}{\cancel{\alpha}} + \frac{2}{\cancel{\alpha^3}} \right]$$

$$\sigma^2 = \frac{1}{\alpha^2}$$

$$\text{S.D.} = \sqrt{\text{Variance}}$$

$$= \sqrt{1/\alpha^2} = \frac{1}{\alpha}$$

NORMAL DISTRIBUTIONS.

The continuous probability distribution having the p.d.f $f(x)$ is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where $-\infty < x < \infty$

and $-\infty < \mu < \infty$ & $\sigma > 0$.

is known as normal distribution

Mean & Variance

$$\text{Mean}(\mu) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{(x-\mu)^2}{2\sigma^2}} x \cdot dx$$

Put $\frac{x-\mu}{\sqrt{2\sigma}} = t$ (1)

$$x - \mu = \sqrt{2\sigma} t$$

$$x = \mu + \sqrt{2\sigma} t$$

diff on both

$$dx = \sqrt{2\sigma} dt$$

where t varies from $-\infty$ to ∞

$$\mu = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} (\mu + \sqrt{2\sigma} t) dt$$

$$\mu = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} (\mu + \sqrt{2\sigma} t) \sqrt{2\sigma} dt$$

$$\mu = \frac{\sqrt{2\sigma}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt + \frac{\sqrt{2\sigma}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} t e^{-t^2} dt$$

even fun odd function

$$\mu = \frac{\mu}{\sqrt{\pi}} + \frac{\sqrt{2\sigma}}{\sqrt{\pi}} \cdot 0$$

$$\mu = \frac{\mu}{\sqrt{\pi}} \cdot 2 \int_0^{\infty} e^{-t^2} dt + 0$$

$$\mu = \frac{2\mu}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2}$$

$$\mu = \mu$$

classmate

by defn of erf func

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

$$\int_a^b f(x) dx =$$

$$\frac{1}{2} \int_0^a f(x) dx = 1886x u$$

even

0 if $f(x)$ is odd

$$\text{variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\text{Put } t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$(x-\mu) = t\sqrt{2}\sigma$$

$$(x-\mu)^2 = t^2 2\sigma^2$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{t^2 - t^2} \sqrt{2} \sigma dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} \frac{t^2 (2t e^{-t^2})}{\frac{1}{t}} dt$$

(By parts integration)

Applying Bernoulli's rule

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} 2t^3 e^{-t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left\{ t(-e^{-t^2}) \right\}_0^{\infty} - \int_0^{\infty} 1(-e^{-t^2}) \cdot dt$$

$$v = \int 2t e^{-t^2} dt$$

$$\text{Put } t^2 = x$$

$$2t dt = dx$$

$$\int e^{-x} dx = -e^{-x}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left[(0-0) + \int_0^{\infty} e^{-t^2} dt \right]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} \right] \text{ By } \Gamma \text{ func}$$

$$\sigma^2 = \sigma^2$$

classmate

$$S.D = \sqrt{\sigma^2} = \underline{\underline{\sigma}}$$

27. Find the value of c such that $f(x) = \begin{cases} -x/6 + c & 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$ is a p.d.f. Also find $P(1 \leq x \leq 2)$

Sol (i) $f(x) \geq 0$ if $c \geq 0$.

(ii) $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_0^3 (-x/6 + c) dx = 1$$

$$\left[\frac{1}{6} \frac{x^2}{2} + cx \right]_0^3 = 1$$

$$\frac{1}{12} (3^2 - 0^2) + c(3 - 0) = 1$$

$$\frac{1}{12} (9) + 3c = 1$$

$$\frac{9}{12} + 3c = 1$$

$$3c = 1 - \frac{9}{12}$$

$$3c = \frac{12 - 9}{12}$$

$$3c = \frac{3}{12}$$

$$c = \frac{1}{12}$$

Now consider $P(1 \leq x \leq 2) = \int_1^2 f(x) dx$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12} \right) dx$$

$$= \left[\frac{1}{6} \cdot \frac{x^2}{2} + \frac{1}{12} x \right]_1^2$$

$$= \frac{1}{12} (2^2 - 1^2) + \frac{1}{12} (2 - 1)$$

$$= \frac{1}{12} (3) + \frac{1}{12} (1)$$

$$= \frac{3}{12} + \frac{1}{12} = \frac{1}{4} + \frac{1}{12}$$

$$= \frac{3+1}{12} = \frac{4}{12} = \frac{1}{3}$$

28. Find the constant k such that $f(x) = \begin{cases} kx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$ is a p.d.f.
Also find $P(1 < x < 2)$

$P(x \leq 1)$ $P(x > 1)$ (iv) find mean (v) Variance

Sol $f(x) \geq 0$ if $k \geq 0$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^3 kx^2 dx = 1$$

$$k \cdot \left[\frac{x^3}{3} \right]_0^3 = 1$$

$$\frac{k}{3} (3^3 - 0^3) = 1 \Rightarrow \frac{k}{3} (27 - 0) = 1 \Rightarrow \frac{27k}{3} = 1$$

$$k = \frac{1}{9}$$

$$\begin{aligned} \text{(i)} \quad P(1 < x < 2) &= \int_1^2 \frac{1}{9} x^2 dx \Rightarrow \frac{1}{9} \left[\frac{x^3}{3} \right]_1^2 \\ &= \frac{1}{9} \cdot \frac{(2^3 - 1^3)}{3} = \frac{1}{9} \left(\frac{7}{3} \right) \\ &= \frac{1}{27} \left(\frac{7}{3} \right) = \frac{7}{27} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad P(x \leq 1) &= \int_0^1 \frac{1}{9} x^2 dx \\ &= \frac{1}{9} \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{27} (1^3 - 0) = \frac{1}{27} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad P(x \geq 1) &= \int_1^3 \frac{1}{9} x^2 dx \Rightarrow \frac{1}{9} \left[\frac{x^3}{3} \right]_1^3 = \frac{1}{27} (27 - 1) \\ &= \frac{26}{27} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad \text{Mean}(\mu) &= \int_0^3 x \cdot f(x) dx \\ &= \int_0^3 x \cdot \frac{1}{9} x^2 dx \\ &= \int_0^3 \frac{1}{9} x^3 dx \\ &= \frac{1}{9} \left[\frac{x^4}{4} \right]_0^3 \Rightarrow \frac{1}{9} \left[\frac{81 - 0}{4} \right] \\ &= \frac{81}{36} = \frac{9}{4} \end{aligned}$$

$$(v) \text{ Variance } (\sigma^2) = \int_0^3 (x - \mu)^2 f(x) dx$$

$$= \int_0^3 x^2 f(x) dx - \mu^2$$

$$= \int_0^3 x^2 \cdot \left(\frac{1}{9}x^2\right) dx - \left(\frac{81}{36}\right)^2$$

$$= \frac{1}{9} \int_0^3 x^4 dx - \left(\frac{81}{36}\right)^2$$

$$= \frac{1}{9} \left[\frac{x^5}{5} \right]_0^3 - \left(\frac{9}{4}\right)^2$$

$$= \frac{1}{9} (243 - 0) - \left(\frac{81}{16}\right)$$

$$= \frac{243}{9} - \frac{81}{16} = \frac{27}{80}$$

29. Find k such that $f(x) = \begin{cases} kxe^{-x} & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ is a p.d.f & also find the mean

$$\frac{k}{e^{-2}} = k \quad \mu = \frac{2e-5}{e-2}$$

30. Find k so that the following function can serve as a p.d.f of a random variable $f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ kxe^{-4x^2} & \text{for } x > 0 \end{cases}$

$$k = 8$$

Exponential

29. ~~The parameter is~~

31. If x is an exponential variety with mean 3, find

(i) $P(x > 1)$

(ii) $P(x < 3)$

Sol The p.d.f of exponential distribution is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & 0 < x < \infty \text{ or } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Given mean $(\mu) = 3 = \frac{1}{\alpha} \Rightarrow \mu = \frac{1}{\alpha}$

$$\Rightarrow \alpha = \frac{1}{3}$$

$$\therefore f(x) = \begin{cases} \frac{1}{3} e^{-1/3 x} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

(i) $P(x > 1) = 1 - P(x \leq 1) \Rightarrow \text{Complement form}$

$$= 1 - \int_0^1 f(x) dx$$

$$= 1 - \int_0^1 \frac{1}{3} e^{-1/3 x} dx$$

$$= 1 - \frac{1}{3} \left\{ \frac{e^{-1/3 x}}{-1/3} \right\}_{x=0}^1$$

$$= 1 + e^{-1/3(1)} - e^0$$

$$= 1 + e^{-1/3} - 1$$

$$= e^{-1/3} = \frac{1}{e^{1/3}}$$

$$= 0.7165$$

$$\begin{aligned} \text{(ii)} P(X < 3) &= \int_0^3 f(x) dx \\ &= \int_0^3 \frac{1}{3} e^{-1/3 x} dx \\ &= \frac{1}{3} \left[\frac{e^{-1/3 x}}{-1/3} \right]_0^3 \\ &= - \left\{ e^{-1/3(3)} - e^{-1/3(0)} \right\} \\ &= -e^{-1} + 1 \\ &= 1 - e^{-1} = 1 - \frac{1}{e} \\ &= \frac{e-1}{e} = 0.6321 \end{aligned}$$

32. The length of telephone (mobile) conversation in a booth has been an exponential distribution & found on an average to be 5 minutes. Find the probability that a random call made from this booth

(i) ends ≤ 5 minutes

(ii) Between 5 & 10 minutes

Sol

we have

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & 0 < x < \infty \quad (x > 0) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Given mean } (\mu) = 5 = \frac{1}{\alpha}$$

$$\alpha = 1/5$$

$$f(x) = \begin{cases} 1/5 e^{-1/5 x} & \text{for } x > 0. \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{(i)} \quad P(X < 5) &= \int_0^5 \frac{1}{5} e^{-1/5 x} dx \\ &= \frac{1}{5} \left[\frac{e^{-1/5 x}}{-1/5} \right]_0^5 \\ &= - \{ e^{-1/5(5)} - e^{-1/5(0)} \} \\ &= - \{ e^{-1} - e^0 \} \\ &= - \frac{-1}{e} + 1 \\ &= 1 - \frac{1}{e} \\ &= \underline{\underline{0.6321}} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad P(5 < X < 10) &= \int_5^{10} \frac{1}{5} e^{-1/5 x} dx \\ &= \frac{1}{5} \left[\frac{e^{-1/5 x}}{-1/5} \right]_5^{10} \\ &= - \{ e^{-1/5(10)} - e^{-1/5(5)} \} \\ &= - (e^{-2} - e^{-1}) \\ &= e^{-2} + e^{-1} \\ &= \underline{\underline{0.2325}} \end{aligned}$$

33. The sales per day in a shop is exponentially distributed with the average sale amounting Rs. 100 & ^{the} net profit 8%. Find the probability that the net profit exceeds Rs. 30 on 2 consecutive days

Let x be the random variable of the sale in a shop. Since x is an exponential variate. The p.d.f

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x > 0. \end{cases}$$

$$\text{Mean}(H) = \frac{1}{\alpha} = \frac{100}{\alpha}$$

$$\alpha = \frac{1}{100} = \underline{\underline{0.01}}$$

$$\text{Hence } f(x) = 0.01 e^{-0.01x} \text{ for } x > 0$$

Let A be the amount for which profit is 8%.

$$\Rightarrow A \cdot \frac{8}{100} = 30$$

$$A = \frac{30 \times 100}{8} = \underline{\underline{375}}$$

Probability of profit exceeding Rs. 30 = $1 - \text{Prob}(\text{profit} \leq \text{Rs. } 30)$

$$= 1 - \text{Prob}(\text{Sales} \leq \text{Rs. } 375)$$

$$= 1 - \int_0^{375} 0.01 e^{-0.01x} dx$$

$$= 1 - 0.01 \left[\frac{e^{-0.01x}}{-0.01} \right]_0^{375} = 1 - [e^{-0.01(375)} - e^0]$$

$$= 1 - [e^{-3.75} - 1] = \underline{\underline{e^{-3.75}}}$$

Probability that profit exceeds ₹300 on a single day

$$= e^{-3.75}$$

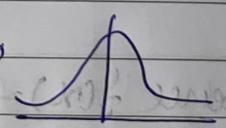
The probability that it repeats on the following day is

$$e^{-3.75} \times e^{-3.75} = e^{-7.5}$$

$$= \underline{\underline{0.00055}}$$

Normal distributions

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz \quad \text{represents the area}$$

under standard normal curve from 0 to z , tabulated values which gives the area for different +ve values of z is readily available.  $z = -\infty \quad z = 0 \quad z = \infty$ such table is called normal probability table. We also bear in mind, the following established result

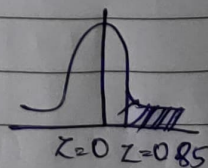
$$\rightarrow \int_{-\infty}^{\infty} \phi(z) dz = 1$$

$$\rightarrow \int_{-\infty}^0 \phi(z) dz = \frac{1}{2} = \int_0^{\infty} \phi(z) dz$$

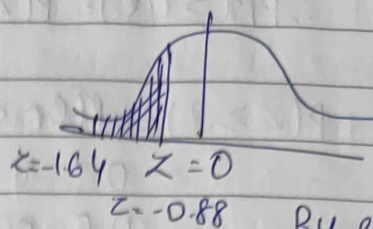
34. Evaluate the following probabilities with the help of normal probability table.

(i) $P(Z \geq 0.85)$ (ii) $P(-1.64 \leq Z \leq -0.88)$

$$\begin{aligned} \text{Sol} \quad P(Z \geq 0.85) &= P(Z \geq 0) - P(Z \leq 0.85) \\ &= 0.5 - \phi(0.85) \\ &= 0.5 - 0.8023 \\ &= \underline{\underline{0.1977}} \end{aligned}$$



$$\begin{aligned} \text{(ii)} \quad & P(-1.64 \leq Z \leq -0.88) \\ &= P(0.88 \leq Z \leq 1.64) \\ &= P(0 \leq Z \leq 1.64) - P(0 \leq Z \leq 0.88) \\ &= \Phi(1.64) - \Phi(0.88) \\ &= 0.4495 - 0.306 \\ &= \underline{\underline{0.1389}} \end{aligned}$$



By symmetry
 $z = 0.88$ to $z = 1.64$

35. If x is a normal variate with mean 30 & S.D = 5. Find the probability that

(i) $26 \leq x \leq 40$ (ii) $x \geq 45$.

Sol we have standard normal variate $z = \frac{x - \mu}{\sigma}$

$$z = \frac{x - 30}{5}$$

(i) To find $P(26 \leq x \leq 40)$.

$$\text{If } x = 26 \text{ then } z = \frac{26 - 30}{5} = -0.8$$

$$\text{If } x = 40 \text{ then } z = \frac{40 - 30}{5} = 2$$

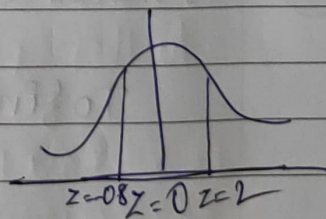
$$\therefore P(26 \leq x \leq 40) = P(-0.8 \leq z \leq 2)$$

$$= P(0 \leq z \leq 0.8) + P(0 \leq z \leq 2)$$

$$= \Phi(0.8) + \Phi(2)$$

$$= 0.2881 + 0.4772$$

$$= \underline{\underline{0.7653}}$$

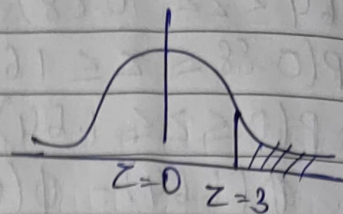


(ii) To find $P(x \geq 45)$

$$\text{If } x = 45 \text{ then } z = \frac{45 - 30}{5} = 3$$

$$\therefore P(X \geq 45) = P(Z \geq 3)$$

$$\begin{aligned} P(Z \geq 3) &= P(0.5 - P(0 \leq Z \leq 3)) \\ &= 0.5 - \phi(3) \\ &= 0.5 - 0.4987 \\ &= \underline{\underline{0.0013}} \end{aligned}$$



36. The marks of 1000 students in an examination follows a normal distribution with mean 70 & std. dev 5. Find the no^r of students whose marks will be

(i) < 65

(iii) Between 65 & 75

(ii) > 75

80

Let x represents the marks of students

By data $\mu = 70$ $\sigma = 5$

$$\text{We S.N.V } Z = \frac{X - \mu}{\sigma} = \frac{X - 70}{5}$$

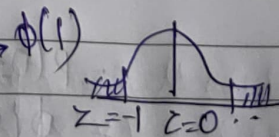
(i) If marks < 45 $\Rightarrow Z = \frac{45 - 70}{5} = -1$

To find $P(Z < -1) = P(Z > 1)$

$$= 0.5 - P(0 \leq Z \leq 1)$$

$$= 0.5 - 0.3413$$

$$= \underline{\underline{0.1587}}$$



11

JOINT PROBABILITY DISTRIBUTION.

We have discussed about the prob distributions associated with a single random variable. The same can be generalised for 2 or more random variables. We discussed the prob distribution associated with 2 random variables referred to as joint probability distributions.

If X & Y are 2 discrete random variables. We define the joint probability of X & Y is defined by $P(X=x, Y=y) = f(x, y)$, where $f(x, y) \geq 0$ and $\sum_x \sum_y f(x, y) = 1$.

It should be observed that $X = x_1, x_2, \dots, x_m$ & $Y = y_1, y_2, \dots, y_n$, then $P(X=x_i, Y=y_j) = f(x_i, y_j) = f_{ij}$. Then the cartesian product of X & Y is given by $X \times Y = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), \dots, (x_m, y_n)\}$.

The joint probability table is

$X \backslash Y$	y_1	y_2	y_3	$\dots$	y_n	sum
x_1	f_{11}	f_{12}	f_{13}	$\dots$	f_{1n}	$f(x_1)$
x_2	f_{21}	f_{22}	f_{23}	$\dots$	f_{2n}	$f(x_2)$
x_3	f_{31}	f_{32}	f_{33}	$\dots$	f_{3n}	$f(x_3)$
x_i	f_{i1}	f_{i2}	f_{i3}	$\dots$	f_{in}	$f(x_i)$
x_m	f_{m1}	f_{m2}	f_{m3}	$\dots$	f_{mn}	$f(x_m)$
sum	$g(y_1)$	$g(y_2)$	$g(y_3)$	$\dots$	$g(y_n)$	1

Add

marginal distr of y

$f(x_1) + f(x_2) + \dots$
Marginal distrib of x

$$\sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) = \sum_{i=1}^m \sum_{j=1}^n J_{ij} = 1$$

EXPECTATIONS, VARIANCE, COVARIANCE & CORRELATIONS

If X be a discrete random variable taking the values $x_1, x_2, \dots, x_n$ having probability $f(x)$, $f(x)$ then the expectation of X is denoted by $E(X)$ or μ_x & it is defined by

$$\mu_x = E_x = \sum_{i=1}^n x_i f(x_i) \text{ or } \sum x f(x)$$

$$\mu_y = E_y = \sum_{j=1}^n y_j f(y_j) \text{ or } \sum y f(y)$$

Variance of X is denoted by $V(X)$ & it is defined by

$$V(X) = \sum_{i=1}^n (x_i - \mu)^2 f(x_i) = E[(X - \mu)^2] \text{ where } \mu \text{ is mean}$$

If X & Y are 2 discrete random variables having the joint prob. func. $f(x, y)$. The expectation of x & y are defined by

$$\mu_x = E(X) = \sum_x \sum_y x f(x, y) = \sum x_i f(x_i)$$

$$\mu_y = E(Y) = \sum_x \sum_y y f(x, y) = \sum y_j f(y_j)$$

$$\text{Further } E(X, Y) = \sum_{ij} x_i y_j J_{ij}$$

If X & Y are the random variables having mean μ_x & μ_y respectively. The covariance of X & Y is denoted by $\text{cov}(X, Y)$ and it is defined as

$$\text{cov}(X, Y) = \sum_i \sum_j (x_i - \mu_x)(y_j - \mu_y) J_{ij}$$

$$= E[(X - \mu_x)(Y - \mu_y)]$$

Equivalently we have

$$\text{cov}(X, Y) = \sum_i \sum_j x_i y_j J_{ij} - \mu_x \mu_y = E(X, Y) - \mu_x \mu_y$$

Correlations of X & Y denoted by $\rho(X, Y)$

$$\rho(X, Y) = \frac{\text{COV}(X, Y)}{\sigma_X \cdot \sigma_Y}$$

Note:- If X & Y are independent ^{random} variable, then

(i) $E(X, Y) = E(X) \cdot E(Y)$

(ii) $\text{COV}(X, Y) = 0 \Rightarrow \rho(X, Y) = 0$

$$\text{COV}(X, X) = E[(X - \mu)^2] = V(X) = \sigma_X^2$$

$$V(X) = E[X^2 - 2X\mu_X + \mu_X^2] \rightarrow (a-b)^2 = a^2 - 2ab + b^2$$

$$= E(X^2) - 2(X \cdot \mu_X) + \mu_X^2 E(1)$$

$$= E(X^2) - 2E(X)E(X) + [E(X)]^2(1)$$

$$V(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

3. The joint distributions of 2 variables X & Y .

	-4	2	7
1	1/8	1/4	1/8 $f(x_i)$ 1/2
5	1/4	1/8	1/8 $f(x_i)$ 1/2

Compute:- $g(y)$ 3/8 $g(y)$ 2/8 $g(y)$ 1/4

(i) $E(X)$ & $E(Y)$ (ii) $E(XY)$ (iii) σ_X & σ_Y (iv) $\text{COV}(X, Y)$ (v) $\rho(X, Y)$

8. The distributions ^{of} X & Y (Marginal).

Distributions of X

Distributions of Y

(i)	x_i	1	5	y_j	-4	2	7
	$f(x_i)$	1/2	1/2	$g(y_j)$	3/8	3/8	1/4

$$\frac{1}{4} + \frac{1}{8} = \frac{2+1}{8} = \frac{3}{8}$$

$$E(X) = \sum x_i f(x_i) = x_1(f(x_1)) + x_2(f(x_2)) = (1)(1/2) + (5)(1/2) = 1/2 + 5/2 = 3$$

$$E(Y) = \sum y_j g(y_j) = y_1 g(y_1) + y_2 g(y_2) + y_3 g(y_3) = (-4)(3/8) + 2(3/8) + 7(1/4) = -1$$

$$(ii) E(X, Y) = \sum x_i y_j f_{ij}$$

$$= (1)(-4)(1/8) + (1)(2)(1/4) + (1)(7)(1/8) + (5)(-4)(1/4) + (5)(2)(1/8) + (5)(7)(1/8)$$

$$= \underline{\underline{3/2}}$$

$$(iii) \sigma_x^2 = E(X^2) - \mu_x^2 \quad \& \quad \sigma_y^2 = E(Y^2) - \mu_y^2$$

$$E(X^2) = \sum x_i^2 f(x_i)$$

$$E(X^2) = (1)(1/2) + (5)^2(1/2)$$

$$= 1(1/2) + (25)(1/2)$$

$$= \underline{\underline{13}}$$

$$E(Y^2) = \sum y_j^2 g(y_j)$$

$$E(Y^2) = (-4)^2(3/8) + (2)^2(3/8) +$$

$$7^2(3/8)$$

$$= 16(3/8) + (4)(3/8) +$$

$$(49)(3/8) = 79/4$$

$$\sigma_x^2 = 13 - (3)^2$$

$$= 13 - 9 = 4$$

$$\sigma_y^2 = (79/4) - 1^2$$

$$= 75/4$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{4} = 2$$

$$\sigma_y = \sqrt{\sigma_y^2} = \sqrt{75/4} = 4.33$$

$$(iv) \text{COV}(X, Y) = E(X, Y) - \mu_x \mu_y$$

$$= 3/2 - (3)(1)$$

$$= 3/2 - 3$$

$$= -3/2$$

$$= \underline{\underline{-1.5}}$$

correlation

$$(v) \rho(X, Y) = \frac{\text{COV}(X, Y)}{\sigma_x \sigma_y} = \frac{-3/2}{2 \sqrt{75/4}} = \frac{-3/2}{2(4.33)} = \underline{\underline{-0.1732}}$$

38. A joint probability distribution is ^{given} by the following table

$X \backslash Y$	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find the variance of X & Y and write the

marginal distributions of $P(X)$ & $P(Y)$

8/8 marginal distributions of X marginal distribution of Y

X	1	3
$P(X)$	0.5	0.5

Y	-3	2	4
$P(Y)$	0.4	0.3	0.3

$$E(X) = \sum x_i f(x_i) \\ = (1)(0.5) + 3(0.5)$$

39. Let X & Y are 2 independent variables. If variance of $(2X - Y) = 6$ & variance of $(X + 2Y) = 9$. Find the variance of X & Y

8/8 To find $\text{Var}(X)$ & $\text{Var}(Y)$
Property: - The variance of a linear combⁿ $aX + bY$ is given by

$$\text{Var}(aX) + \text{Var}(bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$$

We have 2 eq^{ns}

$$\text{Var}(2x - y) = 6$$

$$\text{Var}(x + 2y) = 9$$

We construct the system of eq^{ns}

$$(2)^2 \text{Var}(x) + (-1)^2 \text{Var}(y) = 6$$

$$(1)^2 \text{Var}(x) + (2)^2 \text{Var}(y) = 9$$

$$4 \text{Var}(x) + \text{Var}(y) = 6 \quad \times 1$$

$$\text{Var}(x) + 4 \text{Var}(y) = 9$$

$$\text{Var}(x) = 2$$

$$\text{Var}(y) = 1$$

- 40 A bag contains 2 mangoes & 3 apples & a bag B contains 4 mangoes & 5 apples. 1 fruit is drawn at random from one of the bag. One fruit is drawn at random from one of the bag & it is found to be apple. Find the probability that the apple fruit is drawn from Bag B

Bag A $\rightarrow$ Contains 2M + 3A = 5

Bag B $\rightarrow$ Contains 4M + 5A = 9

$$P(A) = 1/2, P(B) = 1/2$$

$$P(A|A) = \frac{3}{5}$$

$$P(A|B) = \frac{5}{9}$$

From Bayes theorem

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(B) \cdot P(A|B) + P(A) \cdot P(A|A)} = \frac{1/2 (5/9)}{(1/2)(5/9) + (1/2)(3/5)}$$

$$= \frac{5/18}{5/18 + 3/10}$$

$$= \frac{25}{52} \approx 0.4807$$

41. The probability distribution of a random variable X is given below

X	-2	-1	0	1	2
$P(X)$	0.2	0.1	0.3	0.3	0.1

find

(i) $E(X)$ (ii) $V(X)$ (iii) $E(2X-3)$ (iv) $V(2X-3)$

we have $E(X) = \mu_x$

$$\text{Mean } (\mu) = E(X) = \sum x_i P(x_i)$$

$$= (-2)(0.2) + (-1)(0.1) + 0(0.3) + 1(0.3) + 2(0.1)$$

$$= 0$$

$$\text{Variance}(X) = \sum x_i^2 P(x_i) - \mu^2$$

$$= (-2)^2(0.2) + (-1)^2(0.1) + 0^2(0.3) + 1^2(0.3) + 2^2(0.1) - 0^2$$

$$= 0.8 + 0.1 + 0 + 0.3 + 0.4$$

$$= \underline{\underline{1.6}}$$

$E(2X-3)$:- NOTE

$$\text{if } Y = ax + b$$

$$\text{var}(Y) = \text{var}(ax + b)$$

$$= a^2 \text{var}(X)$$

$$E(Y) = E(ax + b) = E(ax) + E(b)$$

$$= aE(X) + b$$

$$E(2X-3)$$

$$\text{Var}(2X-3) = 1.6 - (-3)^2$$

we have

$$= 1 - 9 = -8$$

$$\text{var}(X) = \sum x_i^2 P(x_i) - \mu^2$$

$$= \underline{\underline{1.6}}$$

$$1.6 = \sum x_i^2 P(x_i) - 0^2$$

$$\Rightarrow \sum x_i^2 P(x_i) = 1.6$$

$$E(2X-3) = aE(X) + b$$

$$= 2(0) + (-3)$$

$$= \underline{\underline{-3}}$$

42. For any 2 random variables prove that $E(X+Y) = E(X) + E(Y)$

$$E(X-Y) = E(X) - E(Y)$$

$$\text{Var}(X+Y) = \text{V}(X) + \text{V}(Y) + \text{cov}(X, Y) + \text{cov}(Y, X)$$

$$\text{V}(X-Y) = \text{V}(X) + \text{V}(Y) - \text{cov}(X, Y) - \text{cov}(Y, X)$$

sol $E(X+Y) \Rightarrow$ By definitions of expectations

$$E(X+Y) = \sum_i \sum_j (x_i + y_j) P_{ij}$$

$$= \sum_i \sum_j x_i P_{ij} + \sum_i \sum_j y_j P_{ij}$$

$$= \sum_i x_i \sum_j P_{ij} + \sum_j y_j \sum_i P_{ij}$$

$$= \sum_i x_i P(X=x_i) + \sum_j y_j P(Y=y_j)$$

$$E(X+Y) = E(X) + E(Y)$$

(iii) $V(X+Y) = V(X) + V(Y) + \text{cov}(X, Y) + \text{cov}(Y, X)$

we have $\text{Var}(X) = \text{cov}(X, X) \xrightarrow{\text{By}}$

LHS

$$V(X+Y) = \text{cov}(X+Y, X+Y)$$

$$= \text{cov}(X, X) + \text{cov}(X, Y) + \text{cov}(Y, X) + \text{cov}(Y, Y)$$

$$= V(X) + \text{cov}(X, Y) + \text{cov}(Y, X) + V(Y)$$

$$= V(X) + V(Y) + \text{cov}(X, Y) + \text{cov}(Y, X)$$

(iv) $V(X-Y) = \text{cov}(X-Y, X-Y) \quad \leftarrow \text{Multiplication}$

$$= \text{cov}(X, X) - \text{cov}(X, Y) - \text{cov}(Y, X) + \text{cov}(Y, Y)$$

$$= V(X) - \text{cov}(X, Y) - \text{cov}(Y, X) + V(Y)$$

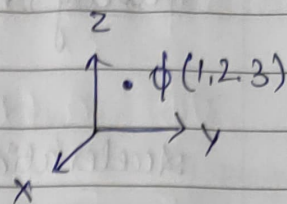
$$= V(X) + V(Y) - \text{cov}(X, Y) - \text{cov}(Y, X)$$

MODULE-3

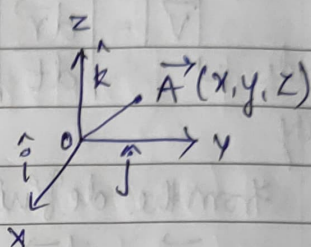
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VECTOR CALCULUS.

SCALAR POINT FUNCTION.

It is defined & denoted by $\phi(x, y, z)$ 

VECTOR POINT FUNCTION

It is defined & denoted by $\vec{A}(x, y, z)$ 

If we assume a constant vector, then we represent

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

GRADIENT (∇), - The gradient is defined & denoted by

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \quad (\text{vector function})$$

If ϕ is a scalar point function, then

$$\nabla \phi = \text{grad } \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

Gradient of vector point function

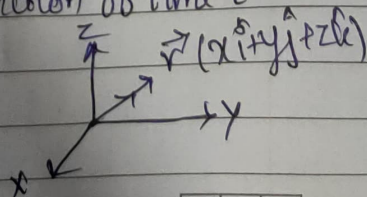
$$\text{Let } \vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\nabla \vec{a} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$$

$$\nabla \cdot \vec{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \rightarrow \text{Scalar}$$

$$\begin{aligned} \hat{i} \cdot \hat{i} &= 1 = \hat{j} \cdot \hat{j} \\ \hat{j} \cdot \hat{j} &= 1 = \hat{k} \cdot \hat{k} \\ \hat{i} \cdot \hat{j} &= 0 = \hat{j} \cdot \hat{k} \end{aligned}$$

VELOCITY & ACCELERATION.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ represents the position vector of a point moving along a curve x, y, z will be a function of time t . $\rightarrow \vec{r}$ is the function of time 't'

$$\text{Velocity } (\vec{v}) = \frac{d\vec{r}}{dt} \rightarrow \frac{\text{displacement}}{\text{time}}$$

$$\text{Acceleration } (\vec{a}) = \frac{d^2\vec{r}}{dt^2}$$

Magnitude of velocity $\rightarrow |\vec{v}| = \left| \frac{d\vec{r}}{dt} \right| = \frac{ds}{dt} = \text{speed}$

From the definition $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \quad (\text{One independent variable time})$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

GRADIENT, DIVERGENCE & CURL OF A VECTOR.

Divergence of vector field:- It is defined & denoted by

$$\text{div}(\vec{A}) = \nabla \cdot \vec{A} \quad (\text{Dot product})$$

$$\text{If } \vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\nabla \cdot \vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

NOTE- If $\text{div}(\vec{A})$ or $\nabla \cdot \vec{A} = 0$ then it is solenoidal

Curl of a vector field:- It is defined & denoted by

$$\text{curl}(\vec{A}) = \nabla \times \vec{A} \quad (\text{Cross product})$$

$$\text{curl } \vec{A} = \nabla \times \vec{A} =$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

If $\nabla \times \vec{A} = 0$ is called irrotational

→ If $\phi(x, y, z)$ is a scalar function, then $\nabla\phi$ is a vector normal to the surface $\phi(x, y, z) = c \rightarrow$ dot product

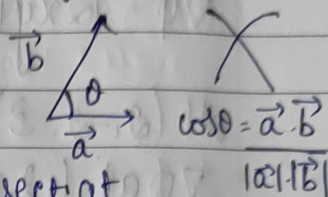
→ The unit vector normal to the surface is defined & denoted by $\hat{n}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{if } \vec{a} = x\hat{i} + y\hat{j} + z\hat{k} \quad \left. \begin{array}{l} \text{Magnitude} \\ \text{is a vector} \end{array} \right\}$$

$$\vec{a} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

→ The angle between 2 surfaces $\phi_1(x, y, z) = c_1$ & $\phi_2(x, y, z) = c_2$ then

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| |\nabla\phi_2|}$$



$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

If $\nabla\phi_1 \cdot \nabla\phi_2 = 0$ then the 2 surfaces intersect at right angle $\odot$ orthogonal

$$\square = \theta = 90^\circ$$

DIRECTIONAL DERIVATIVES

If $\phi(x, y, z)$ is a scalar function & $\vec{a}$ is a given direction, then the directional derivative is $\nabla\phi \cdot \hat{n}$

$$D \cdot D = \nabla\phi \cdot \hat{n} \quad \text{where} \quad \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

COPLANAR VECTORS

If 3 vectors are said to be coplanar, then the scalar triple product should be zero.

Let $\vec{a}, \vec{b}$ & $\vec{c}$ are 3 vectors, then

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

PROBLEMS

1. Find the unit vector normal to the surface with the indicated points following
 $x^2y + 2xy = 4$ at the point $(2, -2, 3)$.

2. Given $\phi(x, y, z) = x^2y + 2xy$

$$\text{Unit vector normal } \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2y+2xyz)\hat{i} + \frac{\partial}{\partial y}(x^2y+2xyz)\hat{j} + \frac{\partial}{\partial z}(x^2y+2xyz)\hat{k}$$

$$\nabla\phi = [y(2x)+2xz(1)]\hat{i} + [x^2(1)+2xz(1)]\hat{j} + \cancel{0} + 2x\hat{k}$$

$$\nabla\phi = \underline{\underline{[2xy+2z]\hat{i} + x^2\hat{j} + 2x\hat{k}}}$$

At (2, -2, 3)

$$\begin{aligned}\nabla\phi \text{ at } (2, -2, 3) &= [2(2)(-2)+2(3)]\hat{i} + 2^2\hat{j} + 2(2)\hat{k} \\ &= -8\hat{i} + 4\hat{j} + 4\hat{k} \\ &= \underline{\underline{-2\hat{i} + 4\hat{j} + 4\hat{k}}}\end{aligned}$$

$$|\nabla\phi| = \sqrt{(-2)^2 + (4)^2 + (4)^2} = \sqrt{4+16+16} = \sqrt{36} = \underline{\underline{6}}$$

$$\hat{n} = \underline{\underline{\frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{6}}}$$

(*) Q. Find the directional derivatives of for the following
(1) $\phi = x^2yz + 4xz^2$ at (3, -2, -1) along $2\hat{i} - \hat{j} - 2\hat{k}$

80 The directional derivative is given by $\nabla\phi \cdot \hat{n}$ where $\hat{n} = \frac{\vec{a}}{|\vec{a}|}$

Given $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$

$$|\vec{a}| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\therefore \hat{n} = \underline{\underline{\frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}}}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x} \hat{i} + \frac{\partial\phi}{\partial y} \hat{j} + \frac{\partial\phi}{\partial z} \hat{k}$$

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = [yz(2x) + 4z^2] \hat{i} + [x^2z + 0] \hat{j} + [x^2y + 4x(2z)] \hat{k}$$

$$= [2xyz + 4z^2] \hat{i} + (x^2z) \hat{j} + (x^2y + 8xz) \hat{k}$$

Now $\nabla\phi$ at $(1, -2, -1)$

$$\nabla\phi = [2(1)(-2)(-1) + 4(-1)^2] \hat{i} + (1(-1)) \hat{j} + (1^2(-2) + 8(1)(-1)) \hat{k}$$

$$= (4 + 4) \hat{i} + (-1) \hat{j} + (-2 - 8) \hat{k}$$

$$= 8 \hat{i} - \hat{j} - 10 \hat{k}$$

The directional derivatives $= \nabla\phi \cdot \hat{n} = 8 \hat{i} - \hat{j} - 10 \hat{k} \cdot \left(\frac{2 \hat{i} - \hat{j} - 2 \hat{k}}{3} \right)$

$$\frac{1}{3} [(8)(2) + (-1)(-1) + (-10)(-2)] = \frac{1}{3} (16 + 1 + 20)$$

$$= \frac{1}{3} (37)$$

$$= \underline{\underline{37/3}}$$

(ii) $\phi = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ along $2\hat{i} - 3\hat{j} + 6\hat{k}$

(iii) $f(x, y, z) = x^2y^2z^2$ at $(1, 1, -1)$ in the direction of the tangent to the curve $x = e^t$, $y = 1 + 2\sin t$, $z = t - \cos t$ where t lies between -1 and 1 . Hence we need to find directional derivative $(\nabla f \cdot \hat{n})$

we have

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$\frac{dy}{dx}$ = To find the slope of tangent

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$$\nabla \phi = 2xy^2z^2\hat{i} + x^2zy^2\hat{j} + x^2y^2(2z)\hat{k}$$

$$= 2xy^2z^2\hat{i} + 2x^2yz^2\hat{j} + 2x^2y^2z\hat{k}$$

$$\nabla \phi \text{ at } (1, 1, -1)$$

$$(\nabla \phi)_{(1, 1, -1)} = 2(1)(1)(-1)^2\hat{i} + 2(1)^2(1)(-1)\hat{j} + 2(1)^2(1)^2(-1)\hat{k}$$

$$= 2\hat{i} + 2\hat{j} - 2\hat{k}$$

In order to find the tangent (direction) vector

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = e^t\hat{i} + (1+2\sin t)\hat{j} + (t - \cos t)\hat{k}$$

Diff w.r.t t

$$\frac{d\vec{r}}{dt} = \text{slope tangent}$$

$$= e^t\hat{i} + 2\cos t\hat{j} + (1 + \sin t)\hat{k}$$

From given data $(1, 1, -1) = P(x, y, z)$

$$e^t = 1 \quad \therefore x = e^t$$

$$\Rightarrow \underline{t = 0}$$

$$e^t = 1$$

$$\log e = \log 1$$

$$t(1) = 0$$

$$t = 0$$

$$t - \cos t$$

$$= 0 - \cos 0$$

$$= 0 - 1 = -1$$

$$1 + 2\sin t = 1$$

$$t - \cos t = 1$$

satisfies all the eqⁿ

$$\therefore \left(\frac{d\vec{r}}{dt}\right)_{t=0} = e^t\hat{i} + 2\cos 0\hat{j} + (1 + \sin 0)\hat{k}$$

$$= 1\hat{i} + 2\hat{j} + 1\hat{k}$$

$$\hat{n} = \frac{d\vec{r}}{dt}$$

$$\hat{i} + 2\hat{j} + \hat{k}$$

$$\frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

$$\left|\frac{d\vec{r}}{dt}\right|$$

$$\sqrt{1^2 + 2^2 + 1^2}$$

$$= \sqrt{4 + 1 + 1}$$

$$\text{Now D.D } \nabla \phi \cdot \hat{n} = \frac{2\hat{i} + 2\hat{j} - 2\hat{k}}{\sqrt{6}} \cdot \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} [(2)(1) + (2)(2) + (-2)(1)]$$

$$= \frac{1}{\sqrt{6}} (4)$$

$$= \underline{\underline{4/\sqrt{6}}}$$

Q. 2

→ directional derivative

Find the rate of change of $\phi(x, y, z) = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point $(1, 1, 1)$

Sol

$$\text{Let } \phi = xyz$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$
$$= yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$(\nabla \phi)_{(1,1,1)} = (1)(1)\hat{i} + (1)(1)\hat{j} + (1)(1)\hat{k}$$
$$= \hat{i} + \hat{j} + \hat{k}$$

~~$$\text{Let } \psi = x^2y + y^2x + yz^2 + zx^2$$~~

~~$$\text{Let } \psi = x^2y + y^2x + yz^2$$~~

~~$$\nabla \psi = (2xy + y^2)\hat{i} + (x^2 + 2yx)\hat{j} + (2yz)\hat{k}$$~~

~~$$= (2xy + y^2)\hat{i} + (x^2 + 2yx)\hat{j} + (2yz)\hat{k}$$~~

~~$$(\nabla \psi)_{(1,1,1)} = (2(1)(1) + 1^2)\hat{i} + (1^2 + 2(1)(1))\hat{j} + (2(1)(1))\hat{k}$$~~

~~$$= 3\hat{i} + 3\hat{j} + 2\hat{k}$$~~

$$\text{Let } \phi = xy^2 + yz^2 + zx^2$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$\nabla \phi = (y^2 + 2xz) \hat{i} + (2xy + z^2) \hat{j} + (2yz + x^2) \hat{k}$$

$$(\nabla \phi)_{(1,1,1)} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\vec{d} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{3(\hat{i} + \hat{j} + \hat{k})}{3\sqrt{3}}$$

$$\sqrt{3^2 + 3^2 + 3^2} = 3\sqrt{3}$$

$$\nabla \phi \cdot \hat{n} = (\hat{i} + \hat{j} + \hat{k}) \cdot \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k})$$

$$= \frac{1}{\sqrt{3}} (1 \times 1 + 1 \times 1 + 1 \times 1) = \frac{3}{\sqrt{3}} = \underline{\underline{\sqrt{3}}}$$

- (V) In which direction the D.D of x^2yz^3 is maximum at $(2, 1, -1)$ and find the magnitude of this maximum

Sol The DD is maximum along the normal vector which is being

$$\phi = x^2yz^3$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$= 2xyz^3 \hat{i} + x^2z^3 \hat{j} + 3x^2yz^2 \hat{k}$$

$$(\nabla \phi)_{(2,1,-1)} = 2(2)(1)(-1)^3 \hat{i} + 2^2(-1)^3 \hat{j} + 3(2)^2(1)(-1)^2 \hat{k}$$

$$= -4\hat{i} - 4\hat{j} + 12\hat{k}$$

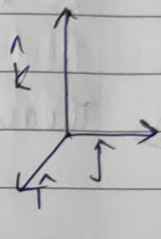
$$\nabla \phi = \sqrt{(-4)^2 + (-4)^2 + (12)^2} = \sqrt{16 + 16 + 144} \\ = 176 = \underline{4\sqrt{11}}$$

(vi) Find the directional derivatives of $\phi = ax^2y^2 + byz + cz^2x^3$ at $(-1, 1, 2)$ has maximum magnitude of 32 units in the direction parallel to y-axis. Find a, b, c.

Maximum D.D is along $\nabla \phi$ & in direction $\parallel$ to y-axis.
The magnitude is given 32 units at $(-1, 1, 2)$

$$\text{Now } \nabla \phi \cdot \hat{j} = 32 \rightarrow \textcircled{1}$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$
$$\phi = ax^2y^2 + byz + cz^2x^3$$



$$\nabla \phi = (ay^2 + cz^2(3x^2))\hat{i} + (ax(2y) + bz)\hat{j} + (by + cx^3(2z))\hat{k}$$

$$(\nabla \phi)_{(-1, 1, 2)} = (a(1)^2 + c(2)^2(3(-1)^2))\hat{i} + (a(-1)(2(1)) + b(2))\hat{j} + (b(1) + c(-1)^3(2(2)))\hat{k}$$
$$= (a + 12c)\hat{i} + (-2a + 2b)\hat{j} + (b - 4c)\hat{k}$$

From $\textcircled{1}$

$$\nabla \phi \cdot \hat{j} = 32$$

$$(-2a + 2b) = 32 \div 2$$

$$-a + b = 16$$

Also since $\nabla \phi$ is $\parallel$ to y axis & $\perp$ to x axis & z axis

$$a + 12c = 0$$

$$b - 4c = 0$$

$$b-a=16$$

On solving

$$a+0b+0c=0$$

$$0a+b-4c=0$$

$$-a+b+0c=0$$

$$\underline{a=-12 \quad b=4 \quad c=1}$$

- 3 Find the angle b/w the surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at $(2, -1, 2)$

sol The angle b/w the surfaces is $\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| \cdot |\nabla \phi_2|}$

$$\text{Let } \phi_1 = x^2 + y^2 + z^2$$

$$\nabla \phi_1 = \frac{\partial \phi_1}{\partial x} \hat{i} + \frac{\partial \phi_1}{\partial y} \hat{j} + \frac{\partial \phi_1}{\partial z} \hat{k}$$

$$= 2x \hat{i} + 2y \hat{j} + 2z \hat{k}$$

$$\text{At } (2, -1, 2)$$

$$(\nabla \phi_1)_{(2, -1, 2)} = 2(2) \hat{i} + 2(-1) \hat{j} + 2(2) \hat{k} = 4 \hat{i} - 2 \hat{j} + 4 \hat{k}$$

$$|\nabla \phi_1| = \sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\phi_2 = x^2 + y^2 - z = 3$$

$$\nabla \phi_2 = \frac{\partial \phi_2}{\partial x} \hat{i} + \frac{\partial \phi_2}{\partial y} \hat{j} + \frac{\partial \phi_2}{\partial z} \hat{k}$$

$$= 2x \hat{i} + 2y \hat{j} + (-1) \hat{k}$$

$$= 2x \hat{i} + 2y \hat{j} - \hat{k}$$

$$\text{At } (2, -1, 2)$$

$$(\nabla \phi_2)_{(2, -1, 2)} = 2(2) \hat{i} + 2(-1) \hat{j} - \hat{k}$$

$$= 4 \hat{i} - 2 \hat{j} - \hat{k}$$

$$|\nabla \phi_2| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{16+4+1} = \sqrt{21}$$

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (4\hat{i} - 2\hat{j} - \hat{k})}{6 \cdot \sqrt{21}} \quad \hat{i} \cdot \hat{i} = 1$$

$$= \frac{16 + 4 - 4}{6\sqrt{21}} = \frac{16}{3\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right)$$

4. Find the angle b/w the normal to the surfaces $xy - z^2$ at the point $(4, 1, 2)$ & $(3, 3, -3)$

Let $\phi = xy - z^2$ &

$\nabla \phi$ is a vector normal to the surface

$$\theta = -0.2132$$

$$\cos \theta = \frac{\nabla \phi_1}{|\nabla \phi_1|}$$

5. Find the value of the constant a & b such that the surfaces $ax^2 - byz = (a+2)x$ & $4x^2y + z^3 = 4$ are orthogonal at the point $(1, -1, 2)$

We need to ensure that the given point lies on both surfaces that means by substituting $(1, -1, 2)$ onto the given eqⁿ, we get

$$a(1)^2 - b(-1)(2) = (a+2)(1)$$

$$a + 2b = a + 2$$

$$2b = 2$$

$$b = 1$$

Also if $(1, -1, 2)$ is substituted onto the LHS of second eqⁿ

$$4x^2y + z^3 = 4, \text{ we get}$$

$$4(1)^2(-1) + (2)^3 = 4$$

$$-4 + 8 = 4$$

$$\underline{4 = 4 = \text{RHS.}}$$

→ that the given point lies on both the surfaces when $b=1$.

In order to find a , we have to use orthogonality condition $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

where $\phi_1 = ax^2 - byz - (a+2)x$ & $\phi_2 = 4x^2y + z^3$

$$\nabla\phi_1 = [2ax - (a+2)]\hat{i} + (-bz)\hat{j} + (-by)\hat{k}$$

$$= [2ax - (a+2)]\hat{i} - bz\hat{j} - by\hat{k}$$

At $(1, -1, 2)$

$$(\nabla\phi_1)_{(1, -1, 2)} = 2a(1) - (a+2)\hat{i} - b(2)\hat{j} - b(-1)\hat{k}$$

$$= (2a - (a+2))\hat{i} - 2b\hat{j} + b\hat{k}$$

$$= (2a - a - 2)\hat{i} - 2b\hat{j} + b\hat{k}$$

$$= (a - 2)\hat{i} - 2b\hat{j} + b\hat{k}$$

$$\nabla\phi_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$

At $(1, -1, 2)$

$$(\nabla\phi_2)_{(1, -1, 2)} = 8(1)(-1)\hat{i} + 4(1)^2\hat{j} + 3(2)^2\hat{k}$$

$$= -8\hat{i} + 4\hat{j} + 12\hat{k}$$

$$\nabla\phi_1 \cdot \nabla\phi_2 = (a-2)(-8) + (-2b)(4) + (b)(12) = 0$$

$$\Rightarrow -8a + 16 - 8b + 12b = 0$$

$$-8a + 4b + 16 = 0$$

But $b=1$

$$-8a + 4 + 16 = 0$$

$$-8a + 20 = 0$$

$$-8a = -20$$

$$a = \frac{-20}{-8} = \frac{5}{2}$$

$$a = \frac{5}{2} \quad \text{and } b = 1$$

8. PROBLEMS ON VELOCITY & ACCELERATIONS.

6. A particle moves along a space curve $\vec{r} = (t^2+t)\hat{i} + (3t-2)\hat{j} + (2t^3-4t^2)\hat{k}$. Find the velocity, acceleration, speed (magnitude of velocity & acceleration) at the time $t=2$ unit.

we have $\vec{r} = (t^2+t)\hat{i} + (3t-2)\hat{j} + (2t^3-4t^2)\hat{k}$

Now diff w.r.t t

$$\vec{v} = \frac{d\vec{r}}{dt} = (2t+1)\hat{i} + (3)\hat{j} + (6t^2-8t)\hat{k} \rightarrow \text{①}$$

At time $t=2$

$$\vec{v} = \left(\frac{d\vec{r}}{dt} \right)_{t=2} = (2(2)+1)\hat{i} + 3\hat{j} + (6(2)^2-8(2))\hat{k}$$

$$(\vec{v})_{t=2} = 5\hat{i} + 3\hat{j} + 8\hat{k}$$

$$|\vec{v}| = \sqrt{5^2 + 3^2 + 8^2} = \sqrt{25 + 9 + 64} = \sqrt{98} = 7\sqrt{2}$$

Diff ① w.r.t t

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 2\hat{i} + 0\hat{j} + (12t-8)\hat{k}$$

$$12(2)-8 = 24-8 = 16$$

$$(\vec{a})_{t=2} = 2\hat{i} + 0\hat{j} + 16\hat{k}$$

$$|\vec{a}| = \sqrt{2^2 + 0^2 + 16^2} = \sqrt{4 + 256} = \sqrt{260}$$

NOTE ① The unit tangent vector $\hat{T}$ is given by $\frac{\vec{v}}{|\vec{v}|} = \hat{T}$

② The tangential component of acceleration is given by $\vec{a} \cdot \hat{T}$ (dot)

③ The normal component of acceleration is given by $|\hat{T} \times \vec{a}|$
 $\rightarrow \perp$ to tangent

④ The component of velocity in the direction of $\vec{n}$ is given by
 $\vec{n} = \vec{v} \cdot \hat{n}$ where $\hat{n} = \frac{\vec{n}}{|\vec{n}|}$

⑤ The component of acceleration in the direction of $\vec{n}$ is given by
 $\vec{n} = \vec{a} \cdot \hat{n}$ where $\hat{n} = \frac{\vec{n}}{|\vec{n}|}$

7. A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t - 5$ where t is the time. Find the component of its velocity & acceleration at time $t = 1$ unit in the direction of $\hat{i} + \hat{j} + 3\hat{k}$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the standard position vector

$$\vec{r} = (t^3 + 1)\hat{i} + t^2\hat{j} + (2t - 5)\hat{k}$$

Diff w.r.t t

$$\vec{v} = \frac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$$

Again diff. w.r.t t

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6t\hat{i} + 2\hat{j} + 0\hat{k} = \frac{d\vec{v}}{dt}$$

At $t = 1$ unit

$$\vec{v}|_{t=1} = 3(1)^2\hat{i} + 2(1)\hat{j} + 2\hat{k} \\ = 3\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{a}|_{t=1} = 6\hat{i} + 2\hat{j} + 0\hat{k}$$

Let $\hat{n}$ be the unit vector in the direction of $\hat{i} + \hat{j} + 3\hat{k}$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{1^2 + 1^2 + 3^2}} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}$$

The component of velocity at $t=1$ in the direction of $\hat{i} + \hat{j} + 3\hat{k}$ is given by

$$= \vec{v} \cdot \hat{n} \\ = 3\hat{i} + 2\hat{j} + 2\hat{k} \cdot \frac{1}{\sqrt{11}} (\hat{i} + \hat{j} + 3\hat{k})$$

$$= \frac{1}{\sqrt{11}} (3 + 2 + 6) = \frac{1}{\sqrt{11}} (11) = \frac{11}{\sqrt{11}} = \underline{\underline{\sqrt{11}}}$$

$\therefore$ comp of velocity = $\sqrt{11}$

The comp of acceleration at $t=1$ = $\vec{a} \cdot \hat{n}$
 $= 6\hat{i} + 2\hat{j} + 0\hat{k} \cdot \frac{1}{\sqrt{11}} (\hat{i} + \hat{j} + 3\hat{k})$

$$= \frac{1}{\sqrt{11}} (6 + 2 + 0)$$

$$= \frac{1}{\sqrt{11}} (8) = \underline{\underline{\frac{8}{\sqrt{11}}}}$$

$\therefore$ comp of acceleration = $\frac{8}{\sqrt{11}}$

8. $x = 2t^2$ $y = t^2 - 4t$ $z = 3t - 5$ at $t=1$ direction $\hat{i} - 3\hat{j} + 2\hat{k}$

9. A particle moves along the curve $\vec{r} = t^2\hat{i} - t^3\hat{j} + t^4\hat{k}$ where t is the time. Find the magnitude of tangential & normal components of its acceleration where $t = 1$

80

we have

$$\vec{r} = t^2 \hat{i} - t^3 \hat{j} + t^4 \hat{k}$$

D. w. r. t. t

$$\vec{v} = \frac{d\vec{r}}{dt} = 2t \hat{i} - 3t^2 \hat{j} + 4t^3 \hat{k}$$

At $t = 1$

$$\vec{v}|_{t=1} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$|\vec{v}| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}$$

Again D. w. r. t. t

$$\frac{d^2\vec{r}}{dt^2} = \vec{a} = 2\hat{i} - 6t\hat{j} + 12t^2\hat{k}$$

At $t = 1$

$$\vec{a}|_{t=1} = 2\hat{i} - 6\hat{j} + 12\hat{k}$$

$$|\vec{a}| = \sqrt{2^2 + (-6)^2 + 12^2} = \sqrt{184}$$

$\therefore$ we need to find the tangential & normal components of acceleration & it is given by $\vec{a} \cdot \hat{t}$ at

$$\text{where } \hat{t} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{2\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{29}} = \frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}$$

$$\Rightarrow \vec{a} \cdot \hat{t} = 2\hat{i} - 6\hat{j} + 12\hat{k} \cdot \frac{1}{\sqrt{29}} (2\hat{i} - 3\hat{j} + 4\hat{k})$$

$$= \frac{1}{\sqrt{29}} (4 + 18 + 48) = \frac{70}{\sqrt{29}}$$

$$\text{Normal component} = |\hat{T} \times \vec{a}|$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{\sqrt{29}} & -\frac{3}{\sqrt{29}} & \frac{4}{\sqrt{29}} \\ 2 & -6 & 12 \end{vmatrix}$$

$$= \hat{i} \left(-\frac{3}{\sqrt{29}} \times 12 + 6 \times \frac{4}{\sqrt{29}} \right) - \hat{j} \left(\frac{2}{\sqrt{29}} \times 12 - 2 \times \frac{4}{\sqrt{29}} \right) +$$

$$\hat{k} \left(\frac{2}{\sqrt{29}} \times (-6) + 2 \left(\frac{3}{\sqrt{29}} \right) \right)$$

$$= \hat{i} \left(\frac{-12}{\sqrt{29}} \right) - \hat{j} \left(\frac{+16}{\sqrt{29}} \right) + \hat{k} \left(\frac{-6}{\sqrt{29}} \right)$$

$$= \frac{-12}{\sqrt{29}} \hat{i} - \frac{16}{\sqrt{29}} \hat{j} - \frac{6}{\sqrt{29}} \hat{k}$$

10. $\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$ where t is the time at $t = 2$

11. Find the tangential & normal component of acceleration at any time t of a particle $x = e^t \cos t$, $y = e^t \sin t$

$$\text{Let } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = e^t \cos t \hat{i} + e^t \sin t \hat{j} + 0\hat{k}$$

D w.r.t t

$$\vec{v} = \frac{d\vec{r}}{dt} = [e^t(-\sin t) + \cos t \cdot e^t]\hat{i} + [e^t(\cos t) + \sin t \cdot e^t]\hat{j} \quad \text{--- (1)}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = e^t [-\sin t + \cos t]\hat{i} + e^t [\cos t + \sin t]\hat{j}$$

$$|\vec{v}| = e^t \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}$$

$$\therefore \sqrt{e^{2t}} = e^t$$

$$\underline{\underline{|\vec{v}| = e^t \sqrt{2}}}$$

$$\vec{a} = \frac{D.W.r.t}{dt^2} = [e^t(-\sin t) - \cos t] \hat{i} + [\cos t - \sin t] e^t \hat{j} + [e^t(-\sin t + \cos t) + (\cos t + \sin t) e^t] \hat{j}$$

$$\vec{a} = [e^t(-2\sin t)] \hat{i} + [e^t(2\cos t)] \hat{j}$$

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{e^t(\cos t - \sin t) \hat{i} + e^t(\cos t + \sin t) \hat{j}}{e^t \sqrt{2}}$$

$$\text{Tangential component} = \vec{a} \cdot \hat{T} = at$$

$$= \frac{1}{e^t \sqrt{2}} \left\{ (e^t)^2 [-2\sin t \cos t + 2\sin^2 t] + (e^t)^2 [2\cos^2 t + 2\cos t \sin t] \right\}$$

$$= \underline{\underline{\sqrt{2} e^t}}$$

$$\text{Normal component} = |\hat{T} \times \vec{a}| = \frac{2e^t}{\sqrt{2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t - \sin t & \cos t + \sin t & 0 \\ -\sin t & \cos t & 0 \end{vmatrix}$$

$$= \underline{\underline{\sqrt{2} e^t}}$$

COPLANAR VECTORS. (Same plane)

The 3 vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

12. Determine the vectors $[2, 4, -1]$, $[8, -10, 5]$ & $[5, -3, 2]$ are coplanar

88] Let $\vec{a} = 2\hat{i} + 4\hat{j} - 1\hat{k}$ & $\vec{b} = 8\hat{i} - 10\hat{j} + 5\hat{k}$ & $\vec{c} = 5\hat{i} - 3\hat{j} + 2\hat{k}$
The vectors are coplanar, then
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0.$$

Now let $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & -10 & 5 \\ 5 & -3 & 2 \end{vmatrix} = \hat{i}(-20+15) - \hat{j}(16-25) + \hat{k}(-24+50)$
$$\vec{b} \times \vec{c} = -5\hat{i} + 9\hat{j} + 26\hat{k}$$

Now consider

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 2\hat{i} + 4\hat{j} - 1\hat{k} \cdot (-5\hat{i} + 9\hat{j} + 26\hat{k})$$

$$\begin{aligned} &= -10 + 36 - 26 \\ &= 0 \end{aligned}$$

13. $P = (3, -1, 4)$, $Q = (6, -4, -8)$ & $S = (-7, -3, 4)$ are coplanar

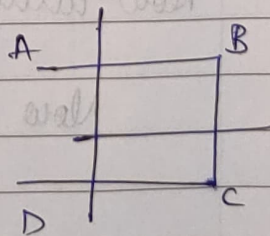
15. Are the 4 points $A(3, 4, -2)$, $B(8, 5, 0)$, $C(1, 10, -6)$ & $D(9, 2, 2)$ are coplanar

88] The given points are A, B, C, D

$$\begin{aligned} \text{We have } \vec{AB} &= \vec{OB} - \vec{OA} \\ &= (5, 1, 2) \end{aligned}$$

$$\vec{BC} = (-7, 5, -6)$$

$$\vec{CD} = (8, -8, -8)$$



$$\vec{AB} \cdot (\vec{BC} \times \vec{CD}) =$$

$$\vec{BC} \times \vec{CD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -7 & 5 & -6 \\ 8 & -8 & 8 \end{vmatrix}$$

$$= \hat{i}(40 - 48) - \hat{j}(-56 + 48) + \hat{k}(56 - 40)$$

$$= -8\hat{i} + 8\hat{j} + 16\hat{k}$$

$$\vec{AB} \cdot (\vec{BC} \times \vec{CD}) = 5\hat{i} + \hat{j} + 2\hat{k} \cdot (-8\hat{i} + 8\hat{j} + 16\hat{k})$$

$$= -40 + 8 + 32$$

$$= \underline{\underline{0}}$$

16. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$, then prove that $\text{grad } u$, $\text{grad } v$, $\text{grad } w$ are coplanar

sol

$$\nabla u = \text{grad } u = \hat{i} + \hat{j} + \hat{k}$$

$$\nabla v = \text{grad } v = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla w = \text{grad } w = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$$

Now calculate $\nabla u \cdot (\nabla v \times \nabla w) = 0$
A.P.T

Now consider $\nabla v \times \nabla w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2x & 2y & 2z \\ (y+z) & (x+z) & (x+y) \end{vmatrix}$

$$= \hat{i}[2y(y+z) - 2z(x+z)] - \hat{j}[2x(x+z) - 2z(y+z)] + \hat{k}[2x(x+z) - 2y(y+z)]$$

$$= \hat{i} [2xy + 2y^2 - 2xz - 2z^2] - \hat{j} [2x^2 + 2xy - 2zy - 2z^2] + \hat{k} [2x^2 + 2xz - 2y^2 - 2yz]$$

Now consider

$$\text{vi. } (\nabla \times \nabla w) = \cancel{2xy + 2y^2} - \cancel{2xz - 2z^2} - \cancel{2x^2} - \cancel{2xy} + \cancel{2zy} + \cancel{2z^2} + \underline{\underline{2x^2 + 2xz - 2y^2 - 2yz}}$$

$$= 0$$

USEFUL

GRADIENTS

Useful identities for computing gradients

$$1. \frac{\partial [f(x)^T]}{\partial x} = \left[\frac{\partial f(x)}{\partial x} \right]^T$$

tr = trace

$$2. \frac{\partial [\text{tr}(f(x))]}{\partial x} = \text{tr} \left(\frac{\partial f(x)}{\partial x} \right)$$

$$3. \frac{\partial (\det f(x))}{\partial x} = \det f(x) \cdot \text{tr} \left(f(x)^{-1} \cdot \frac{\partial f(x)}{\partial x} \right)$$

$$4. \frac{\partial [f(x)^{-1}]}{\partial x} = -f(x)^{-1} \frac{\partial f(x)}{\partial x} f(x)^{-1}$$

$$5. \frac{\partial (a^T x^{-1} b)}{\partial x} = -(x^{-1})^T a b^T (x^{-1})^T$$

$$6. \frac{\partial (x^T a)}{\partial x} = a^T$$

$$7. \frac{\partial (a^T x)}{\partial x} = a^T$$

$$8. \frac{\partial}{\partial x} (a^T x b) = a b^T$$

$$9. \frac{\partial}{\partial x} (x^T B x) = x^T (B + B^T)$$

I GRADIENT OF A VECTOR VALUED FUNCTION OF SCALARS & VECTORS

$$\textcircled{1}. \left(\frac{\partial \vec{a}}{\partial x} \right) = \frac{\partial a_i}{\partial x} \Rightarrow \text{i-th component of } \frac{\partial \vec{a}}{\partial x} \text{ is } \frac{\partial a_i}{\partial x}$$

$\uparrow$ vector
 $\downarrow$ scalar

Ex:- Let $\vec{a} = \begin{bmatrix} x^2 \\ x^3 \\ x^5 \end{bmatrix}$ then $\frac{\partial \vec{a}}{\partial x} = \begin{bmatrix} 2x \\ 3x^2 \\ 5x^4 \end{bmatrix} \rightarrow \text{Scalar}$

$$\textcircled{2}. \left(\frac{\partial x}{\partial \vec{a}} \right)_i = \frac{\partial x}{\partial a_i}$$

$\uparrow$ scalar
 $\downarrow$ vector
 $\rightarrow$ vector function

Ex:- $f(x_1, x_2, x_3) = x_1 x_2 x_3^2$

$$\frac{df}{dx} = \frac{\partial f}{\partial \vec{x}} = \nabla f \rightarrow \text{gradient of } f$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{bmatrix}$$

$$\nabla f = [x_2 x_3^2, x_1 x_3^2, x_1 x_2 (2x_3)]$$

vectors can be represented as matrix also

DATE

II VECTORS & VECTORS.

$\vec{a}$ & $\vec{b}$ are vectors

① $\left(\frac{\partial \vec{a}}{\partial \vec{b}}\right)_{ij} = \frac{\partial a_i}{\partial b_j} \rightarrow$ Matrix rep. (2^{nd} order tensors)
 (i^{th} component of $\vec{a}$ differentiations w.r.t j^{th} component of $\vec{b}$)

Ex:- $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\frac{\partial \vec{a}}{\partial \vec{b}} = \begin{bmatrix} \frac{\partial a_1}{\partial b_1} & \frac{\partial a_1}{\partial b_2} \\ \frac{\partial a_2}{\partial b_1} & \frac{\partial a_2}{\partial b_2} \end{bmatrix}$$

$$\text{Jacobian (J)} = \frac{\partial (a_1, a_2)}{\partial (b_1, b_2)} = \begin{bmatrix} \frac{\partial a_1}{\partial b_1} & \frac{\partial a_1}{\partial b_2} \\ \frac{\partial a_2}{\partial b_1} & \frac{\partial a_2}{\partial b_2} \end{bmatrix}$$

$$\therefore J \begin{pmatrix} u, v \\ x, y \end{pmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

② $\frac{\partial}{\partial x} (\vec{x} \cdot \vec{a}) = \frac{\partial}{\partial x} (\vec{x}^T \vec{a}) = \frac{\partial}{\partial x} (\vec{a}^T \vec{x}) = \vec{a}$

Ex:- Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$
 $\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$

$$\vec{x} \cdot \vec{a} = a_1 x_1 + a_2 x_2 + a_3 x_3$$

$$\frac{\partial}{\partial x} (\vec{x} \cdot \vec{a}) = \begin{bmatrix} \frac{\partial}{\partial x_1} (\vec{x} \cdot \vec{a}) \\ \frac{\partial}{\partial x_2} (\vec{x} \cdot \vec{a}) \\ \frac{\partial}{\partial x_3} (\vec{x} \cdot \vec{a}) \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \vec{a}$$

VECTOR VALUED FUNCTION $\rightarrow$ $\{$ of mapping from

It is represented as $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ where $n \geq 1$ & $m \geq 1$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ \vdots \\ f_n(x) \end{bmatrix} \in \mathbb{R}^m \text{ where } x = [x_1, x_2, x_3, \dots, x_n]^T \in \mathbb{R}^n$$

~~Ex~~ Ex:- $f(x, y) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

17. Define gradient of a vector valued function

The vector valued function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ has vector of functions $[f_1, f_2, \dots, f_m]$, $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$ that map onto $\mathbb{R}$.

Partial derivatives of a vector valued functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with respect to $x_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, then

$$\frac{\partial f}{\partial x} = \frac{\partial f(x)}{\partial x_i} = \begin{bmatrix} \frac{\partial f_1}{\partial x_i} \\ \frac{\partial f_2}{\partial x_i} \\ \vdots \\ \frac{\partial f_n}{\partial x_i} \end{bmatrix}$$

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} \\ \vdots \\ \frac{\partial f_1}{\partial x_n} \end{bmatrix}$$

JACOBIAN.

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ mapping vector $x_i \in \mathbb{R}^n$ onto a scalar posses Jacobian that is a row vector $(m \times n)$ matrix

$$\therefore J = \nabla_x f = \frac{df(x)}{dx} = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} & \dots & \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \dots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \dots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \dots & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix}$$

This is called the Jacobian, where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $J(e_j) = \frac{\partial f}{\partial x_j}$

Gradient of Matrix

$$\frac{\partial}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} \end{bmatrix}$$

Ex: ① $y = 4x_3 + 3x_2 + 2x_1 + x_0$ w.r.t scalar w.r.t
 $x = \begin{bmatrix} x_0 & x_2 \\ x_1 & x_3 \end{bmatrix}$ $\rightarrow$ scalar matrix
 $\rightarrow$ vector

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

② vector w.r.t matrix

$$\text{let } \vec{y} = (e^{x_0 x_1}, e^{x_2 x_3})$$

$$\frac{\partial \vec{y}}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} \\ \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_3} \end{bmatrix} = \begin{bmatrix} e^{x_0 x_1(x_1)} & e^{x_2 x_3(x_2)} \\ e^{x_0 x_1(x_0)} & e^{x_2 x_3(x_2)} \end{bmatrix}$$

18. Find the jacobian of $f(x_1, x_2) = (-2x_1 + x_2, x_1 + x_2)$

Sol. Let $\vec{x} = x_1, x_2$

$$f_1(x) = -2x_1 + x_2$$

$$f_2(x) = x_1 + x_2$$

$$J \left\{ \begin{matrix} f_1 \\ f_2 \end{matrix} \right\} \frac{\partial (f_1, f_2)}{\partial (x_1, x_2)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$|J| = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -2 - 1 = -3 \rightarrow \text{Scaling factor}$$

19. Find the jacobian of $f(x_1, x_2, x_3) = (3x_1 x_2 + x_1 x_3, 2x_1 - 4x_3^2, (x_1 + x_2 + x_3)^2)$ at $(2, 1, -1)$

Sol.

$$\vec{x} = x_1, x_2, x_3$$

$$f_1(x) = 3x_1 x_2 + x_1 x_3$$

$$f_2(x) = 2x_1 - 4x_3^2$$

$$f_3(x) = (x_1 + x_2 + x_3)^2$$

$$J \frac{\partial (b_1, b_2, b_3)}{\partial (x_1, x_2, x_3)} = \begin{bmatrix} \frac{\partial b_1}{\partial x_1} & \frac{\partial b_1}{\partial x_2} & \frac{\partial b_1}{\partial x_3} \\ \frac{\partial b_2}{\partial x_1} & \frac{\partial b_2}{\partial x_2} & \frac{\partial b_2}{\partial x_3} \\ \frac{\partial b_3}{\partial x_1} & \frac{\partial b_3}{\partial x_2} & \frac{\partial b_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} 3x_2 + x_3 & 3x_1 & x_1 \\ 2 & 0 & -8x_3 \\ 2(x_1 + x_2 + x_3)(1) & 2(x_1 + x_2 + x_3) & 2(x_1 + x_2 + x_3)(1) \end{bmatrix}$$

(1)

at (2, 1, 1)

 $x_1 \ x_2 \ x_3$

$$= \begin{bmatrix} 2 & 6 & 2 \\ 2 & 0 & 8 \\ 4 & 4 & 4 \end{bmatrix}$$

$$|J| = \begin{vmatrix} 2 & 6 & 2 \\ 2 & 0 & 8 \\ 4 & 4 & 4 \end{vmatrix}$$

$$= 2(0 - 32) - 6(8 - 32) + 2(8 - 0)$$

$$= -64 + 144 + 16$$

$$= 96$$

$$= 2(0 - 32) - 6(8 - 32) + 2(8 - 0)$$

$$= -64 + 144 + 16$$

$$= 96$$

20. Compute the gradient of 'h' w.r.t t for the function $h: \mathbb{R} \rightarrow \mathbb{R}$
 $h(t) = (\log)(t)$ with $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $g: \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $f(x) = e^{xy^2}$
 $x = \begin{bmatrix} x \\ y \end{bmatrix}$, $g(t) = \begin{bmatrix} t \cos t \\ t \sin t \end{bmatrix}$ $x = t \cos t$, $y = t \sin t$

$$\frac{dy}{dt} = \{-t \sin t + \cos t\}$$

$$dy/dt = \{t \cos t + \sin t\}$$

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sol

since $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ & $g: \mathbb{R} \rightarrow \mathbb{R}^2$

$$\frac{\partial f}{\partial x} \in \mathbb{R}^{1 \times 2} \quad \frac{\partial g}{\partial t} \in \mathbb{R}^{2 \times 1}$$

Gradient computed by applying chain rule

$$f(x \rightarrow y) \rightarrow t$$

Chain rule is

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

~~$$\frac{df}{dt} = e^{xy^2} \{t(-\sin t) + \cos t(1)\} + e^{xy^2}$$~~

$$\frac{df}{dt} = e^{xy^2} \cdot y^2 \{t(-\sin t) + \cos t(1)\} + e^{xy^2} \cdot x(2y) \{t \cos t + \sin t(1)\}$$

$$= e^{xy^2} \cdot y^2 \{t(-\sin t) + \cos t\} + e^{xy^2} \cdot 2xy \{t \cos t + \sin t\}$$

Q21

Consider a function $h: \mathbb{R} \rightarrow \mathbb{R}$ $h(t) = (f \circ g)(t)$ with

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$ $g: \mathbb{R} \rightarrow \mathbb{R}^3$ defined by $f(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_3 x_1$

and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $g(t) = \begin{bmatrix} t \cos t \\ t \sin t \\ t \end{bmatrix}$

sol

$$f \rightarrow (x_1, x_2, x_3) \rightarrow t$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \frac{\partial f}{\partial x_3} \frac{dx_3}{dt}$$

$$x_1 = t \cos t$$

$$x_2 = t \sin t$$

$$x_3 = t$$

$$\frac{dx_1}{dt} = t(-\sin t) + \cos t$$

$$\frac{dx_2}{dt} = t \cos t + \sin t$$

$$\frac{dx_3}{dt} = 1$$

$$= (x_2 + x_3) \frac{d}{dt} (t \cos t) + (x_1 + x_3) \frac{d}{dt} (t \sin t) + (x_2 + x_1) \frac{d}{dt} (t)$$

$$\frac{dB}{dt} = (x_2 + x_3) \{ t(-\sin t) + \cos t \} + (x_1 + x_3) (t \cos t + \sin t) + (x_2 + x_1) (1)$$

Q22 Find the gradient of $f(x_1, x_2) = x_1^2 x_2 + x_1 x_2^3$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad f_1(x) = x_1^2 x_2$$

$$f_2(x) = x_1 x_2^3$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt}$$

$$\leq 2x_1 x_2 + x_2^3$$

$$\frac{\partial f}{\partial x_1} = 2x_1 x_2 + x_2^2$$

$$\frac{\partial f}{\partial x_2} = x_1^2 + 3x_1 x_2^2$$

$$\text{Gradient of } f(x_1, x_2) = \nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 x_2 + x_2^2 \\ x_1^2 + 3x_1 x_2^2 \end{bmatrix}$$

Q23 Find the gradient of $f(x, y) = x^2 \sin y$

$$\frac{\partial f}{\partial x} = 2x \sin y \quad \frac{\partial f}{\partial y} = x^2 (\cos y)$$

$$\text{Gradient of } f(x, y) = \nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \sin y \\ x^2 \cos y \end{bmatrix}$$

24. Find the gradient of $f(x, y) = 3x^2 + 2y + 6$ at $(1, -1)$

Sol $\frac{\partial f}{\partial x} = 6x$ $\frac{\partial f}{\partial y} = 2$

Gradient $f(x, y) = \nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 6x \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 6(1) \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

25. Find the gradient of $f(x, y, z) = x^2 y^3 z^4$

Sol $\frac{\partial f}{\partial x} = 2xy^3z^4$ $\frac{\partial f}{\partial y} = x^2(3y^2)z^4$ $\frac{\partial f}{\partial z} = x^2y^3(4z^3)$
 $= 2xy^3z^4$ $= 3x^2y^2z^4$ $= 4x^2y^3z^3$

$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2xy^3z^4 \\ 3x^2y^2z^4 \\ 4x^2y^3z^3 \end{bmatrix}$

26. Find the gradient of $f(x, y, z) = e^x \sin y \log z$

Sol $\frac{\partial f}{\partial x} = e^x \sin y \log z$ $\frac{\partial f}{\partial y} = e^x \cos y \log z$ $\frac{\partial f}{\partial z} = e^x \sin y \left(\frac{1}{z}\right)$

$$\nabla f(x, y, z) = \begin{bmatrix} e^x \sin y \log z \\ e^x \cos y \log z \\ e^x \sin y \left(\frac{1}{z}\right) \end{bmatrix}$$

27. If $f(x, y) = x^2 + 2y$; where $x = \sin t$ & $y = \cos t$ then find $\frac{df}{dt}$

Sol Chain rule
 $f \rightarrow (x, y) \rightarrow t$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{df}{dt} = 2x \cos t + 2(-\sin t)$$

28. Compute the derivative of the function $h(x) = g[f(x)]$ where $g[f(x)] = [f(x)]^4$ & $f(x) = 2x+1$

Sol We have by chain rule

$$\begin{aligned} h' &= \frac{dh}{dx} = \frac{d}{dx} [g(f(x))] \\ &= \frac{d}{dx} [f(x)]^4 \\ &= 4[f(x)]^3 f'(x) \end{aligned}$$

but $f(x) = 2x+1$

& $f'(x) = 2$

$$= 4[8(x)]^3(2)$$

$$= 8(2x+1)^3$$

29. Find $h'(x)$ where $h(x) = f(g(\sin 4x))$

80

By chain rule

$$h'(x) = \frac{dh}{dx}$$

$$= \frac{d}{dx} [f(g(\sin 4x))]$$

$$= f'(g(\sin 4x)) g'(\sin 4x) \times \underline{\underline{4 \cos 4x}}$$

BACKPROPAGATIONS-

In machine learning, backpropagation is a gradient estimation method used to train neural network models.

The gradient estimate is used by the optimisation algorithm to compute the network parameter updates.

It is just a way of propagating the total loss back into the neural network, to know how much of the loss every node is responsible for and subsequently updating the weights in a way that minimises the loss by giving the nodes with higher error rates lower weights & vice versa.

It is an iterative algorithm to minimise the cost function by determining which weights & biases should be adjusted. During every Epoch (iteration), the model learns by adjusting the weights & biases to minimise the loss by moving down towards the gradient of the error.

Weights & Biases.

These are the neural network parameters that simplify machine learning data identification. The weights & biases developed how a neural network propels data flow forward through the network.

Weights refers to connection management between 2 basic units within a neural network

Biases ~~are~~ in neural network are additional crucial units in sending data to the correct end

Terms related to weights & Biases.

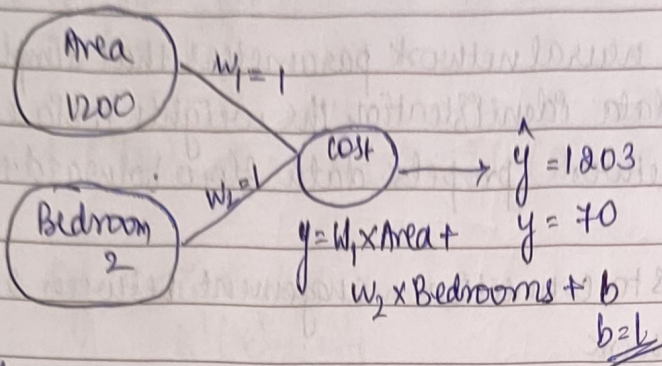
Neuron :- Basic units of the neural network containing individual data features

Layers :- A collection of unrelated neurons (input layer) that connect to another set of neurons. This continues until the final layer of neurons (output layer) is reached. Weights affect the neuron connections b/w the layers.

Hidden layers :- layers between input layers & output layers. This is where artificial neurons take in a set of weighted ~~neuro~~ inputs & produce an output through an activation function.

~~For~~ To understand the backpropagation by chain rule, we take an ex.

Area	Bedroom	Price
1200	2	70
1500	3	120
2500	4	230
1100	3	105
1300	2	90
900	1	55



Initially we take $w_1 = 1$ & $w_2 = 1$ first sample. The predicted $\hat{y}$ value is

$$\hat{y} = 1 \times 1200 + 1 \times 2 + 1 = 1203$$

Predicted value $\hat{y} = 1203$ & compare with actual value of $y = 70$

$$\therefore \text{Error} = y - \hat{y} = 70 - 1203 = -1133$$

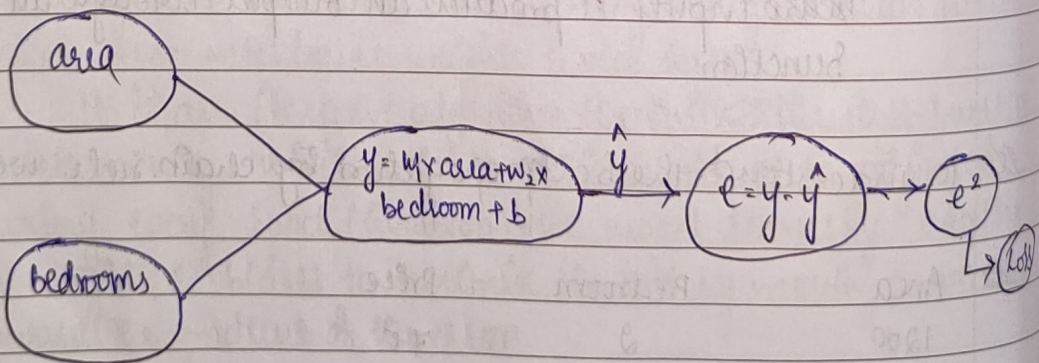
$$\text{Squared Error} = 1283689$$

Take all the errors & sum it, we get

$$\text{Total squared error} = \text{Sq. Error}_1 + \dots + \text{Sq. Error}_6$$

$$\text{Mean Squared Error} = \frac{(\text{Sq. Error}_1) + \dots + \text{Sq. Error}_6}{6}$$

$$\therefore \text{Loss} = \text{Mean Squared Error}$$



we get loss is more, we adjust the weights to reduce the loss.
Adjust weights

$$w_1 = w - \text{something}$$

$$w_1 = w_1 - \text{learning rate} \times \frac{\partial}{\partial w_1}$$

$$\text{III} \quad w_2 = w_2 - \text{learning rate} \times \frac{\partial}{\partial w_2}$$

$$\text{Bias } b = b - \text{learning rate} \times \frac{\partial}{\partial b}$$

random value

$$w_1 = 1 - 0.2 = 0.8$$

$$w_2 = 1 - 0.3 = 0.7$$

$$b = 0 - 0.2 = -0.2$$

all our training samples with new weights and keep on doing this until we get optimal value (maximum or minimum). That is called Gradient Descent (Global minima). $[L = e^2 \quad ge = y - \hat{y}]$

$$\text{mathematically } \frac{\partial L}{\partial e} = 2e, \quad \frac{\partial e}{\partial \hat{y}} = -1, \quad \frac{\partial \hat{y}}{\partial w_1} = \text{area}$$

$$\hat{y} = w_1 \text{area} + w_2 \times b + 1$$

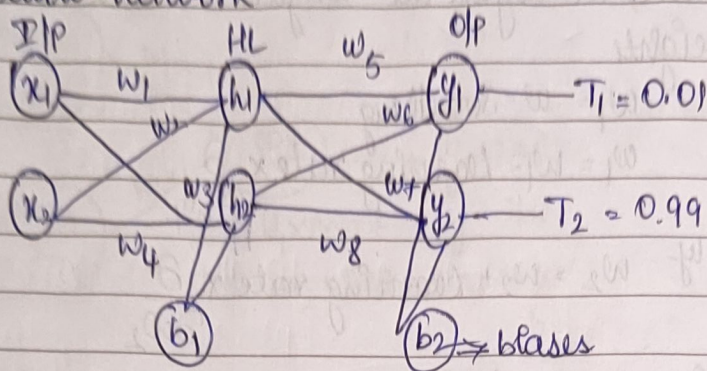
$$\frac{\partial L}{\partial w_1} = 0, \text{ loss w.r.t } w_1 \text{ (Partial derivative of loss w.r.t } w_1).$$

$$\frac{\partial L}{\partial w_1} = \frac{\partial L}{\partial e} \cdot \frac{\partial e}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial w_1} \rightarrow \text{chain rule.}$$

$$\frac{\partial L}{\partial w_1} = 2e \cdot (-1) \cdot \text{area} = -2e \cdot \text{area}$$

for a given change in w_1

30 In neural network



88 We have $H_1 = w_1 x_1 + w_2 x_2 + b_1$

Activation function is sigmoid = $\frac{1}{1+e^{-x}}$

$$\text{output } H_1 = \frac{1}{1+e^{-x}}$$

Initially Ex:- $w_1 = 0.15$ $w_2 = 0.20$ $w_3 = 0.25$ $w_4 = 0.30$ $w_5 = 0.40$
 $w_6 = 0.45$ $w_7 = 0.50$ $w_8 = 0.55$

$$x_1 = 0.05 \quad x_2 = 0.10 \quad b_1 = 0.35 \quad b_2 = 0.60$$

$$T_1 = 0.01 \quad T_2 = 0.99$$

Forward pass:-

$$H_1 = (0.15)(0.05) + (0.20)(0.10) + 0.35$$

$$= 0.3775$$

$$\text{O/P of } H_1 = \frac{1}{1+e^{-x}} = \frac{1}{1+e^{-0.3775}} = 0.593269992$$

In same way

$$H_2 = x_1 w_3 + x_2 w_4 + b_2$$

$$= (0.05)(0.25) + (0.10)(0.30) + 0.60$$

$$= 0.3925$$

$$O/P H_2 = \frac{1}{1+e^{-x}} = \frac{1}{1+e^{-0.3925}} = \underline{\underline{0.596884378}}$$

Calculate y_1

$$\begin{aligned} y_1 &= \cancel{H_1} O/P H_1 w_5 + O/P H_2 w_6 + b_2 \\ &= 0.593269992 (0.40) + 0.596884378 (0.45) + 0.60 \\ &= \underline{\underline{1.105905967}} \end{aligned}$$

$$O/P y_1 = \frac{1}{1+e^{-x}} = \underline{\underline{0.75136507}}$$

$$\text{Calculate } y_1 = 0.77292845$$

total
Calculating error:-

$$\begin{aligned} E_{\text{total}} &= \sum \frac{1}{2} (\text{target} - \text{output})^2 \\ &= \cancel{0.01 - 0.9} \frac{1}{2} (T_1 - \text{output } y_1)^2 + \frac{1}{2} (T_2 - \text{output } y_1)^2 \\ &= \frac{1}{2} (0.01 - 0.75136507)^2 + \frac{1}{2} (0.99 - 0.77292845)^2 \\ &= \underline{\underline{0.298371109}} \end{aligned}$$

Now backpropagate to update weights

$$\text{Error at } w_5 = \frac{\partial E_{\text{total}}}{\partial w_5}$$

By chain rule

$$\frac{\partial E_{\text{total}}}{\partial w_5} = \frac{\partial E_{\text{total}}}{\partial (\text{out } y_1)} \cdot \frac{\partial (\text{out } y_1)}{\partial y_1} \cdot \frac{\partial y_1}{\partial w_5} \rightarrow \textcircled{1}$$

As direct diff is not possible

E_{total} is not does not have w_5 . Therefore we apply chain rule

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We have $E_{total} = \frac{1}{2} (T_1 - out_{y_1})^2 + \frac{1}{2} (T_2 - out_{y_2})^2$

Now $\frac{\partial E_{total}}{\partial out_{y_1}} = \frac{1}{2} \times 2 (T_1 - out_{y_1})' (-1) + 0$

$\frac{\partial E_{total}}{\partial out_{y_1}} = -10.01 - 0.75136507$
 $\frac{\partial E_{total}}{\partial out_{y_1}} = \underline{\underline{0.74136507}}$

Now $out_{y_1} = \frac{1}{1 + e^{-y_1}}$

$\frac{\partial out_{y_1}}{\partial y_1} = - (1 + e^{-y_1})^{-2} e^{-y_1} (-1)$
 $= \frac{1}{(1 + e^{-y_1})^2} \cdot e^{-y_1}$

$\frac{\partial out_{y_1}}{\partial y_1} = (out_{y_1})^2 \left(\frac{1 - out_{y_1}}{out_{y_1}} \right)$

$\frac{\partial out_{y_1}}{\partial y_1} = out_{y_1} (1 - out_{y_1})$

$\frac{\partial out_{y_1}}{\partial y_1} = 0.75136507 (1 - 0.75136507)$
 $\frac{\partial out_{y_1}}{\partial y_1} = \underline{\underline{0.186815602}}$

Now $out_{y_1} = H(w_5 + H(w_6 + b_2))$
 D. p wrt w_5

$\frac{\partial y_1}{\partial w_5} = H_1$

$\frac{\partial y_1}{\partial w_5}$

$\frac{\partial y_1}{\partial w_5} = 0.593269992$

$\frac{\partial y_1}{\partial w_5}$

① becomes

$$\frac{\partial E_{total}}{\partial w_5} = 0.74136507 \times 0.186815602 \times 0.593269992$$

$$\frac{\partial w_5}{\partial w_5} = 0.082167041$$

↓
change in w_5

Updating w_5 Learning rate should be assumed from 0 to 1 only

$$w_5 = w_5 - \text{Learning rate}(\eta) \times \frac{\partial E_{total}}{\partial w_5}$$

$$= 0.4 - 0.5 \times 0.082167041$$

$$= 0.35891648$$

III

$$w_6 = 0.408666180$$

$$w_7 = 0.51301270$$

$$w_8 = 0.56137021$$

} New updated weights

Now at hidden layer, we update the w_1, w_2, w_3, w_4

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial outH_1} \cdot \frac{\partial outH_1}{\partial H_1} \cdot \frac{\partial H_1}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial outH_1} = \frac{\partial E_1}{\partial outH_1} + \frac{\partial E_{total}}{\partial outH_2}$$

$$\frac{\partial E_1}{\partial outH_1} = \frac{\partial E_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial outH_1}$$

$$\frac{\partial E_1}{\partial y_1} = \frac{\partial E_1}{\partial outy_1} \cdot \frac{\partial outy_1}{\partial y_1}$$

updating $w_1 = 0.149780716$

$$w_2 = 0.19956143$$

$$w_3 = 0.24975114$$

$$w_4 = 0.29950229$$

OPTIMIZATIONS.

Optimisation

Constrained



Lagrange's multipliers

Unconstrained

↓
Gradient
Descent
method

↓
Steepest
Descent
method

↓
Cauchy's
method

CONSTRAINED OPTIMISATION.

It involves some conditions like $g(x) \leq b$ & $a < x < b$ and $f(x)$ can be maximised or minimised.

UNCONSTRAINED OPTIMIZATIONS.

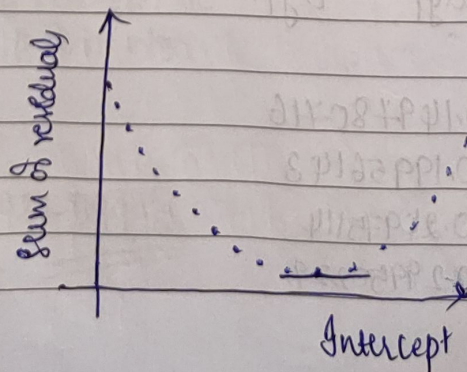
Gradient Descent:

In statistics, machine learning & other data science fields, we optimize a lot of stuff.

When we fit a line with linear regression, we optimize the intercept & slope.

NOTE In machine learning, the sum of the several residuals is a type of loss functions.

If we plot the sum of the several residual value for the intercept 0 as the initial approximation.



Gradient descent, will be used to find where sum of squares residual is lowest [slope near to zero are lower]

Definition:- An algorithm to minimize a function by optimising its parameters

$$J(m, c) = \sum_{i=1}^n (y_i - (mx_i + c))^2 \quad y = mx + c$$

Here m & c are parameters (scalars). This functions to be minimised for the optimal value of m & c .

Ex:-

x	y
1	2
3	4

Initial assumptions

$$c=0 \text{ \& } m=1$$

~~$$J(m, c) = (2 - (m(1) + c))^2 + (4 - (m(3) + c))^2$$~~

$$J(m, c) = (2 - (m(1) + 0))^2 + (4 - (m(3) + 0))^2$$

$$\frac{\partial J}{\partial c} = 2[2 - (m + c)](-1) + 2[4 - (3m + c)](-1)$$

$$\frac{\partial J}{\partial c} = -2[2 - (1 + 0)] - 2[4 - 3 - 0](-1)$$

$$= -2[2 - 1] - 2[4 - 3]$$

$$= -2(1) - 2(1)$$

$$= -2 - 2 = -4$$

$$LR = 0.001$$

(step size) → assumed

$$c_{\text{new}} = c_{\text{old}} - LR \times \frac{\partial J}{\partial c}$$

$$= 0 - 0.001(-4)$$

$$= 0.004$$

$$\begin{aligned}\frac{\partial J}{\partial m} &= 2[2 - (m+c)](-1) + 2[4 - (3m+c)](-3) \\ &= -2[2 - (1+0)] + 2[4 - (3(1)+0)](-3) \\ &= -2[2-1] - 6[4-3] \\ &= -2[1] - 6[1] \\ &= -2 - 6 = \underline{\underline{-8}}\end{aligned}$$

$$\begin{aligned}m_{\text{new}} &= m_{\text{old}} - LR \times \frac{\partial J}{\partial m} \\ &= 1 - 0.001(-8) \\ &= \underline{\underline{1.008}}\end{aligned}$$

1st iteration $m=1.008$ $c=0.004$

Entire process will repeat till we reach the minimum cost function.
How does the algorithm know that it has residual minimum cost function? (When the slope is nearer to zero)

⊗ Gradient Descent:-

An algorithm to minimise a function by optimising its parameter

Algorithm:- It is a set of instruction that work on some given data in a particular way

working rule

Iteration 1:-

Step 1:- Find the search direction $S_i = -\nabla f_i = -\nabla f(x_i)$

Step 2:- Determine the optimal length λ_i in the direction of S_i and let $x_{i+1} = x_i + \lambda_i S_i$ where

$$\lambda_i = \frac{S_i^T \times S_i}{S_i^T \cdot H S_i}$$

where $H =$ Hessian matrix $= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$ $\frac{\partial^2 f}{\partial x_1 \partial x_2}$ mixed partial derivative

Step 3:- Test the optimality for the new point

until we

stop the iteration, nearly reach 0 or equal to 0.

1. Use gradient descent or steepest descent method for $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$, starting from the point $x_1 = (0, 0)$

we have $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$

$$\nabla f = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix} \rightarrow \textcircled{1}$$

Hessian matrix $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$

$$H = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_1^2} = +4$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 2 - 1 + 2x_1 + 2x_2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 2$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} \rightarrow 1 + 4x_1 + 2x_2$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2$$

Ist iteration

Step 1:- Find the search direction S_1 at $x_1(0,0)$

$$\therefore S_1 = -\nabla f(x_1) = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}}$$

Step 2:- Compute λ_1 at x_1

$$\lambda_1 = \frac{S_1^T \cdot S_1}{S_1^T \cdot H S_1} = \frac{\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 1+1 \end{bmatrix}}{\begin{bmatrix} -4+2 & -2+2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \frac{2}{\begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$= \frac{2}{(2+0)} = \frac{2}{2} = \underline{\underline{1}}$$

$$\lambda_1 = \underline{\underline{1}}$$

New point $x_{i+1} = x_i + \lambda_i S_i$
 $i=1$

$$x_2 = x_1 + \lambda_1 S_1$$
$$= (0,0) + (1) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_2 = \underline{\underline{(-1,1)}}$$

IInd iteration

$$S_2 = -\nabla f(x_2) = \cancel{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \cancel{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

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$$= - \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}$$

Step 2:- Compute λ_2 at x_2

$$\lambda_2 = \frac{S_2^T \cdot S_2}{S_2^T \cdot H S_2} = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 1+1 \end{bmatrix}}{\begin{bmatrix} 4+2 & 2+2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{2}{\begin{bmatrix} 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{2}{\begin{bmatrix} 6+4 \end{bmatrix}} = \frac{2}{10} = \underline{\underline{\frac{1}{5}}}$$

$$\lambda_2 = \frac{1}{5}$$

 $i=2$

$$\begin{aligned} x_3 &= x_2 + \lambda_2 S_2 \\ &= (-1 \ 1) + \left(\frac{1}{5}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (-1 \ 1) + \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix} \\ &= \left[-\frac{4}{5}, \frac{6}{5}\right]^T = \begin{bmatrix} -0.8 \\ 1.2 \end{bmatrix} \end{aligned}$$

$$\text{Step 3:- } \nabla f(x_3) = \begin{pmatrix} 0.2 \\ -0.2 \end{pmatrix} + \underline{\underline{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}}$$

Iteration 3

$$\text{Step 1:- } S_3 = -\nabla f(x_3) = -\begin{bmatrix} +0.2 \\ -0.2 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

Step 2 To find λ_3 at x_3

$$\lambda_3 = \frac{S_3^T S_3}{S_3^T H S_3}$$

$$\lambda_3 = \frac{[-0.2, 0.2] \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}}{[-0.2 \ 0.2] \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 0 & 0 \\ 0.4 & 0.4 \end{bmatrix}}{\begin{bmatrix} -0.8 + 0.4 & -0.4 + 0.4 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 0.8 \end{bmatrix}}{\begin{bmatrix} -0.4 & 0 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 0.8 \end{bmatrix}}{\begin{bmatrix} 0.8 + 0 \end{bmatrix}} = \underline{\underline{1}}$$

$$\underline{\underline{\lambda_3 = 1}}$$

New point $x_4 = x_3 + \lambda_3 S_3$

$$= \begin{bmatrix} +0.8 \\ -0.2 \end{bmatrix} + 1 \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2 \\ +0.2 \end{bmatrix} + \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} -1 \\ 1.4 \end{bmatrix}}}$$

$$\text{Step 3: } -\nabla f(x_4) = \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix}$$

Iteration 4

$$S_4 = -\nabla f(x_4) = -\begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$

$$\lambda_4 = \frac{S_4^T S_4}{S_4^T H S_4}$$

$$= \frac{\begin{bmatrix} 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}}{\begin{bmatrix} 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}}$$

$$= \frac{[0.04 + 0.04]}{[0.8 + 0.4, 0.4 + 0.4] \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}}$$

$$= \frac{0.08}{[0.2 \ 0.8] \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}} = \frac{0.8}{0.24 + 0.16}$$

$$= \frac{0.08}{0.4}$$

$$\lambda_4 = \underline{\underline{0.2}}$$

$$x_5 = x_4 + \lambda_4 S_4$$

$$= \begin{bmatrix} -1 \\ 1.4 \end{bmatrix} + 0.2 \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0.04 \\ 0.04 \end{bmatrix}$$

$$= \begin{bmatrix} -1 + 0.04 \\ 1.4 + 0.04 \end{bmatrix} = \begin{bmatrix} -0.96 \\ 1.44 \end{bmatrix}$$

$$\nabla f(x_5) = \begin{bmatrix} 0.04 \\ -0.04 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

NOTE:- ① Steepest Descent method is an iterative method also known as gradient descent method

② Finding minimum value is the target

Q. Use steepest descent to find minimum of $f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2$. Show that error does not exceed 0.05 for the function. Initial approximation $x_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$

Sol Given $x_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} \rightarrow \textcircled{1}$$

$$\frac{\partial f}{\partial x_1} = 2x_1 - x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 2$$

D.p. wrt x_2

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -1$$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x_2} = -x_1 + 2x_2$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2$$

D.p. wrt x_1

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -1$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}$$

1st Iteration

To find search direction S_1 at $x_1 (1, 1/2)$

$2 - 1/2$

Step 1

$$S_1 = -\nabla f(x_1) = - \begin{bmatrix} 2(1) - 1/2 \\ -1 + 2(1/2) \end{bmatrix} = - \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$$

$$\frac{4-1}{2} = \frac{3}{2}$$

$$= \underline{\underline{\begin{bmatrix} -1.5 \\ 0 \end{bmatrix}}}$$

Step 2 Compute λ_1 at x_1

$$\lambda_1 = \frac{S_1^T S_1}{S_1^T H S_1} = \frac{\begin{bmatrix} -3/2 & 0 \end{bmatrix} \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3/2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}}$$

$$\lambda_1 = \frac{\begin{bmatrix} 9/4 + 0 \end{bmatrix}}{\begin{bmatrix} 3+0 & 3/2+0 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}} = \frac{\begin{bmatrix} 9/4 \end{bmatrix}}{\begin{bmatrix} -3 & 3/2 \end{bmatrix} \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}}$$

$$= \frac{9/4}{-9/2 + 0} = \frac{9/4 \times 2}{9/1} = \underline{\underline{\frac{1}{2}}}$$

new point

$$x_2 = x_1 + \lambda_1 S_1$$

$$= (1, 1/2) + \frac{1}{2} \begin{bmatrix} -3/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} + 1/2 \begin{bmatrix} -3/2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -3/4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}}}$$

Step 3. Check the optimum value

$$\nabla f(x_2) = \begin{bmatrix} 2(1/4) - 1/2 \\ -1/4 + 2(1/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} \neq \underline{\underline{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}}$$

2nd iteration

$$S_2 = -\nabla f(x_2) = -\begin{bmatrix} 0 \\ 3/4 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ -3/4 \end{bmatrix}}}$$

$$x_2 \leftarrow \lambda_2 = \frac{S_2^T \cdot S_2}{S_2^T \cdot H S_2}$$

$$= \begin{bmatrix} 0 & -3/4 \end{bmatrix} \begin{bmatrix} 0 \\ -3/4 \end{bmatrix}$$

$$\begin{aligned} & -3/4 \times -3/4 \\ & = 9/16 \end{aligned}$$

$$\begin{bmatrix} 0 & -3/4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -3/4 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 0 & 9/16 \end{bmatrix}}{\begin{bmatrix} 0 & 3/4 & 0 & -3/2 \end{bmatrix} \begin{bmatrix} 0 \\ -3/4 \end{bmatrix}} = \frac{9/16}{0 + 9/8} = \frac{9/16 \times 2}{9/8} = \underline{\underline{\frac{1}{2}}}$$

$$x_3 = x_2 + \lambda_2 S_2$$

$$= \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ -3/4 \end{bmatrix}$$

$$-3/4 \times \frac{1}{2}$$

$$= -3/8$$

$$= \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3/8 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1/4 \\ 1/8 \end{bmatrix}}}$$

$$-1/4 + 2/8 = 0$$

$$1/2 - 1/8 = \frac{4-1}{8} = \frac{3}{8}$$

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$$\nabla f(x_3) = \begin{bmatrix} 2(1/4) - 1/8 \\ -1/4 + 2(1/8) \end{bmatrix} = \begin{bmatrix} 3/8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0 \end{bmatrix} \Rightarrow \underline{\underline{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}}$$

3. $f(x_1, x_2) = 2x_1^2 + 2x_1x_2 + x_2^2 + x_1 - x_2$ $x_1 = [0, 0]$

4. $f(x_1, x_2) = 4x_1^2 + 6x_2^2 - 8x_1x_2$ with $x_1 = (1, 1)$ Min 3 iterations

5. For a quadratic function in 2 Dimension $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^T$

$$\begin{bmatrix} 2 & 1 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} \text{ with gradient } \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 20 \end{bmatrix} -$$

$\begin{bmatrix} 5 \\ 3 \end{bmatrix}^T$ Initial location $x_0 = [-3, -1]^T$ & $\gamma = 0.085$. Apply gradient descent algorithm which converges to minimal value perform 2 iterations.

6. Minimise by using gradient descent $f(x_1, x_2, x_3) = x_1^2 + x_1(1-x_2) + x_2^2 - x_2x_3 + x_3^2 + x_3$ with $x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

we have $f(x_1, x_2, x_3) = x_1^2 + x_1(1-x_2) + x_2^2 - x_2x_3 + x_3^2 + x_3$
 $= x_1^2 + x_1 - x_1x_2 + x_2^2 - x_2x_3 + x_3^2 + x_3$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 + 1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 + 1 \end{bmatrix}$$

Hessian matrix $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix} \lambda_1 = 1/2 \quad \lambda_2 = 1/2$$
$$S_2 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}$$
$$\lambda_1 = 1/2 \quad \lambda_2 = 1/2$$

CONSTRAINED OPTIMIZATIONS & LAGRANGE'S MULTIPLIERS

Maximise

7. ~~Max~~ $Z = 6x^2 + 5xy + 2y^2$ subjected to $2x + y = 96$

Sol we have

Step 1:- Set the constraint equal to 0

$$96 - 2x - y = 0$$

Step 2:- write down the Lagrangian functions

$$L: (6x^2 + 5xy + 2y^2) + \lambda (96 - 2x - y) = 0$$

$\downarrow$ Utility / objective func $\rightarrow$ Lagrangian parameter $\rightarrow$ Budget constraint

Step 3:- Take the first order partial derivatives of L w.r.t x, y and λ .

$$\frac{\partial L}{\partial x} = 12x + 5y - 2\lambda = 0 \rightarrow \textcircled{1}$$

$$\frac{\partial L}{\partial y} = 5x + 4y - \lambda = 0 \rightarrow \textcircled{2}$$

$$\frac{\partial L}{\partial \lambda} = 96 - 2x - y = 0 \rightarrow \textcircled{3}$$

Set of partial derivatives are equated to zero

④ solve the system of eqn

from (2)

$$\lambda = 5x + 4y$$

$$(3) \quad y = 96 - 2x$$

Eqn (1) becomes

$$12x + 5(96 - 2x) - 2(5x + 4y) = 0$$

$$12x + 480 - 10x - 10x - 8(96 - 2x) = 0$$

$$-12x - 8x + 480 - 768 + 16x = 0$$

$$+ 8x - 288 = 0$$

$$8x = 288$$

$$x = 36$$

Substituting x in y

$$y = 96 - 2(36)$$

$$y = 24$$

Substituting x & y in λ

$$\lambda = 5(36) + 4(24)$$

$$= 276$$

The function is maximum at $x = 36$ & $y = 24$ The maximum value at $x = 36$ & $y = 24$ is 132488. Maximise the function $f(x, y) = 2xy$ subjected to constraints $x + y = 6$

sol we have

step 1 :- Set the constraint to zero

$$6 - x - y = 0$$

(2) Lagrangian function

$$f(x, y, \lambda) = L: 2xy + \lambda(6 - x - y)$$

③ Take first order partial derivative

$$\frac{\partial L}{\partial x} = 2y - \lambda = 0 \rightarrow \textcircled{1}$$

$$\frac{\partial L}{\partial y} = 2x - \lambda = 0 \rightarrow \textcircled{2}$$

$$\frac{\partial L}{\partial \lambda} = 6 - x - y = 0 \rightarrow \textcircled{3}$$

From ②

$$\lambda = 2x$$

From ③

$$y = 6 - x$$

Eqⁿ ① becomes

$$2(6 - x) - \lambda = 0$$

$$12 - 2x - \lambda = 0$$

From ②

$$12 - 2x - 2x = 0$$

$$12 - 4x = 0$$

$$4x = 12$$

$$x = 3$$

$$\lambda = 2x$$

$$\lambda = 2(3)$$

$$\lambda = 6$$

Substing x in ①

$$y = 6 - x$$

$$y = 6 - 3$$

$$y = 3$$

The function is maximum at $x = 3$ & $y = 3$

The maximum value is $z = 18$

Maximum of $f(x, y) = 2xy$ subjected to $x + y = 6$ is 18 at

9. Consider the production function $f(x, y) = 60x^{3/4}y^{1/4}$ suppose that labour costs \$100 per unit and capital costs \$200 per unit. Assume that \$30,000 is available to spend on cost. How many units of labour & capital will maximise production?

Here the constraint is $100x + 200y = 30000$
 Given $f(x, y, z) = 60x^{3/4}y^{1/4}$

Lagrangian $L = f(x, y, z) = (60x^{3/4}y^{1/4}) + \lambda(30000 - 100x - 200y)$

D.p.w.r.t x, y, λ are equal to 0

$$\frac{\partial L}{\partial x} = 60 \cdot \frac{3}{4} x^{3/4-1} y^{1/4} - 100\lambda = 0$$

$$= 45 x^{-1/4} y^{1/4} - 100\lambda = 0 \rightarrow (1)$$

$$\frac{\partial L}{\partial y} = 60 x^{3/4} \left(\frac{1}{4} y^{1/4-1} \right) - 200\lambda = 0$$

$$= 15 x^{3/4} y^{-3/4} - 200\lambda = 0 \rightarrow (2)$$

$$\frac{\partial L}{\partial \lambda} = 30000 - 100x - 200y = 0 \rightarrow (3)$$

Eqn divide by 200

~~From~~

$$30000 - x - 2y = 0 \rightarrow (3)$$

From (1)

$$100\lambda = 45 x^{-1/4} y^{1/4}$$

$$\lambda = \frac{9}{20} x^{-1/4} y^{1/4}$$

$\hookrightarrow (4)$

From (2)

$$200\lambda = 15 x^{3/4} y^{-3/4}$$

$$\lambda = \frac{3}{40} x^{3/4} y^{-3/4}$$

$\hookrightarrow (5)$

Equating (4) & (5)

$$\frac{9}{20} x^{-1/4} y^{1/4} = \frac{3}{40} x^{3/4} y^{-3/4}$$

× both side by $x^{1/4}$

$$\frac{9}{20} x^{1/4} x^{-1/4} y^{1/4} = \frac{3}{40} x^{3/4} x^{1/4} y^{-3/4}$$

$$\frac{9}{20} y^{1/4} = \frac{3}{40} x y^{-3/4}$$

Multiplied by $y^{3/4}$ on both sides

$$\frac{9}{20} y^{1/4} y^{3/4} = \frac{3}{40} x y^{3/4} y^{-3/4}$$

$$\frac{9^3}{20} y = \frac{3}{40} x$$

$$\underline{\underline{x = 6y}} \quad \textcircled{2} \quad \underline{\underline{y = \frac{1}{6}x}}$$

Sub $y = \frac{1}{6}x$ in (3)

$$300 - x - 2\left(\frac{1}{6}x\right) = 0$$

$$300 - x - \frac{1}{3}x = 0$$

$$\underline{900 - 3x - x = 0}$$

$$3$$

$$\frac{900}{3} - \frac{4x}{3} = 0$$

$$\frac{900 - 4x}{3} = 0$$

$$4x = 900$$

$$x = \frac{900}{4}$$

$$x = \underline{\underline{225}}$$

Substituting $x = 225$ in $y = \frac{1}{6}x$

$$y = \frac{1}{6}(225)$$

$$y = \frac{75}{2}$$

$$y = \underline{\underline{37.5}}$$

Substituting x & y in ①

$$\lambda = \frac{9}{20} (225)^{1/4} (37.5)^{1/4}$$

$$= \underline{\underline{0.2875}}$$

10. ~~opt~~ Minimise $f(x, y) = x^2 + y^2$ subject to $x + y = 10$.

Let constraint = 0.

$$10 - x - y = 0$$

$$f(x, y, \lambda) = L: (x^2 + y^2) + \lambda(10 - x - y) = 0$$

$$\frac{\partial L}{\partial x} = 2x - \lambda = 0 \rightarrow \textcircled{1}$$

$$\frac{\partial L}{\partial y} = 2y - \lambda = 0 \rightarrow \textcircled{2}$$

$$\frac{\partial L}{\partial \lambda} = 10 - x - y = 0 \rightarrow \textcircled{3}$$

From ②

$$\lambda = 2y$$

From ③

$$y = 10 - x$$

Eqⁿ ① becomes

$$2(10-x) - \lambda = 0$$

$$20 - 2x - \lambda = 0$$

from ②

$$20 - 2x - \lambda = 0$$

$$20 - 4x = 0$$

$$20 = 4x$$

$$x = \frac{20}{4}$$

$$\underline{\underline{x = 5}}$$

sub $x = 5$ in ③

$$y = 10 - 5$$

$$\underline{\underline{y = 5}}$$

sub $x = 5$ & $y = 5$ in ①

$$\lambda = 2(5)$$

$$\underline{\underline{\lambda = 10}}$$

$\therefore$ Value is maximum at $x = 5$ & $y = 5$

$$\text{Value is } (5)^2 + (5)^2 = 25 + 25 = \underline{\underline{50}}$$

Q. 11 If $x + y + z = a$ show that the maximum value of $x^m y^n z^p$ is $\frac{a^{m+n+p}}{(m+n+p)^{m+n+p}}$

Sol.

$$\text{Let } f(x, y, z) = x^m y^n z^p$$

Now consider $f(x, y, z, \lambda) = x^m y^n z^p + \lambda(a - x - y - z)$
D p wrt x, y, z & λ and equate to 0

$$\therefore \frac{\partial f}{\partial x} = mx^{m-1} y^n z^p - \lambda = 0 \rightarrow \text{①}$$

$$\frac{\partial \delta}{\partial y} = x^m n y^{n-1} z^p = \lambda \rightarrow (2)$$

$$\frac{\partial \delta}{\partial z} = x^m y^n p z^{p-1} = \lambda \rightarrow (3)$$

$$\frac{\partial \delta}{\partial x} = a - x - y - z \rightarrow (4)$$

Now from (1) (2) & (3)

$$m x^{m-1} y^n z^p = n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$$

LHS equal,
equates RHS.

from 1st & 2nd ratio

$$m x^{m-1} y^n z^p = n x^m y^{n-1} z^p$$

$$m x^{-1} = n y^{-1}$$

$$\Rightarrow m y = n x$$

from 2nd & 3rd ratio.

$$n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$$

$$n y^{-1} = p z^{-1}$$

$$n z = p y$$

from 1st & 3rd

$$\Rightarrow y = \frac{n x}{m}$$

$$z = \frac{p y}{n}$$

Now consider $x + y + z = a$

$$x + \left(\frac{n x}{m}\right) + \left(\frac{p y}{n}\right) = a$$

$$x + \frac{n x}{m} + \frac{p}{n} \left(\frac{n x}{m}\right) = a$$

$$x + \cancel{\frac{n x}{m}} x + \frac{n x}{m} + \frac{p x}{m} = a$$

$$m x + n x + p x = m a$$

$$x (m + n + p) = m a$$

$$x = \frac{m a}{m + n + p}$$

$$y = \frac{na}{m+n+p}$$

$$z = \frac{pa}{m+n+p}$$

The maximum value of $x^m y^n z^p$ is given by

$$\left(\frac{ma}{m+n+p}\right)^m \cdot \left(\frac{na}{m+n+p}\right)^n \cdot \left(\frac{pa}{m+n+p}\right)^p$$

$$\Rightarrow m^m n^n p^p \cdot \left(\frac{a}{m+n+p}\right)^{m+n+p}$$

12. A rectangular box open at top is to have a volume of 32 cubic feet. Find its dimension if the total surface area is minimum.

Sol. Let x, y, z respectively be the length, breadth & height of rectangular box.

Let V be the volume and S be the surface area

$$V = xyz = 32 \text{ cubic units}$$

with respect to total surface area = $2(xy + yz + zx)$

Since the box is open at top the surface area is given by

$$S = 2(xy + yz + zx) - xy$$

$$S = 2xy + 2yz + 2zx$$

To find x, y, z such that 'S' is minimum subject to the condition
 $xyz = 32$

Let $f(x, y, z, \lambda) = xy + 2yz + 2zx + \lambda(32 - xyz) \rightarrow (1)$
D. p wrt x, y, z & λ

$$\frac{\partial f}{\partial x} = (y + 2z) - yz\lambda = 0 \rightarrow (2)$$

$$\frac{\partial f}{\partial y} = (x + 2z) - xz\lambda = 0 \rightarrow (3)$$

$$\frac{\partial f}{\partial z} = (2y + 2x) - \lambda xy = 0 \rightarrow (4)$$

$$\frac{\partial f}{\partial \lambda} = 32 - xyz = 0 \rightarrow (5)$$

from (2)

$$y + 2z = \lambda yz$$
$$\lambda = \frac{y + 2z}{yz}$$

from (3)

$$\lambda = \frac{x + 2z}{xz}$$

from (4)

$$\lambda = \frac{2y + 2x}{xy}$$

$\Rightarrow$ from (2), (3) & (4)

$$\frac{y + 2z}{yz} = \frac{x + 2z}{xz} = \frac{2y + 2x}{xy}$$

from 1st & 2nd ratio.

$$\frac{y + 2z}{yz} = \frac{x + 2z}{xz}$$

$$x(y + 2z) = y(x + 2z)$$
$$xy + 2xz = xy + 2yz$$

CM

$$2xz = 2yz$$

$$x = y$$

from ① & ③ ratio

$$\frac{x+2z}{xz} = \frac{2x+2y}{xy}$$

CM

$$xy + 2xz = 2xz + 2yz$$

$$\underline{\underline{y = 2z}}$$

from ① & ③ ratio

$$\frac{y+2z}{yz} = \frac{2x+2y}{xy}$$

CM

$$xy + 2xz = 2xz + 2zy$$

$$\underline{\underline{x = 2z}}$$

Substituting $y = 2z$ & $x = y = 2z$ & $z = \frac{x}{2}$

$$\text{Now } xyz = 32$$

$$x(2)\left(\frac{x}{2}\right) = 32$$

$$\frac{x^3}{2} = 32$$

$$x^3 = 64$$

$$\underline{\underline{x = 4}} \Rightarrow \underline{\underline{y = 4}} \quad \underline{\underline{z = 2}}$$

$$929.0304 \text{ cm}^3$$

$$432 \text{ sq cm}$$

$$0.3171 \text{ cubic feet}$$

x^2
All degrees 2 - Quadratic

DATE

13. Using Lagrange's multipliers find the dimensions of the rectangular box open at the top whose maximum capacity with ~~max~~ surface area is 432 sq cm

14. Minimise $f(x, y, z) = x^2 + y^2$ and constraint is $x + y + z = 10$

15. Produce TV sets at the factories a, b. Suppose x TV's are produced at A and y TV's are produced at B. The cost is given by $6x^2 + 12y^2$. If you must produce 90 TV sets is the not that should be maximum
 $x + y = 90 \Rightarrow$ constraint

CONVEX OPTIMIZATIONS.

* we have Quadratic form

* Nature / ~~definitions~~ definiteness of Hessian matrix

All diagonal Positive

Negative

Semi +ve
+ve semi

Semi -ve
-ve semi

QUADRATIC FORM

A homogenous expressions of 2nd degree in any no of variables is called quadratic form

Ex:- (i) $3x^2 + 2xy + 4y^2$

(ii) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - x_2x_3 + 6x_3x_1$

In terms of matrix (Hessian matrix & its nature).

~~the~~ In terms of matrix, we get

$$X^T A X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X^T A X = a_{11} x_1^2 + a_{21} x_1 x_2 + a_{12} x_1 x_2 + a_{22} x_2^2$$

NOTE - The symmetric matrix A associated with quadratic form is given by

$$A = \begin{bmatrix} \text{coeff } x_1^2 & \frac{1}{2} \text{coeff } x_1 x_2 \\ \frac{1}{2} \text{coeff } x_2 x_1 & \text{coeff } x_2^2 \end{bmatrix}$$

Examples:- 1. Find the Hessian matrix of $x^2 + 2x_2^2 + 3x_3^2 + 4x_1 x_2 + 6x_1 x_3 + 8x_2 x_3$

Sol

Matrix $A =$	$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix}$	diagonal x^2 rest elements $\frac{1}{2}$ times
--------------	---	--

Note - For a quadratic function $f(x) = X^T A X$, we have $H = 2A$

2. Quadratic form

To find Hessian matrix for $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1 x_2 - 2x_1 x_3 + 4x_2 x_3$

Sol

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} \text{coeff } x_1^2 & \frac{1}{2} \text{coeff } x_1 x_2 & \frac{1}{2} \text{coeff } x_1 x_3 \\ \frac{1}{2} \text{coeff } x_2 x_1 & \text{coeff } x_2^2 & \frac{1}{2} \text{coeff } x_2 x_3 \\ \frac{1}{2} \text{coeff } x_3 x_1 & \frac{1}{2} \text{coeff } x_3 x_2 & \text{coeff } x_3^2 \end{bmatrix}$$

Hessian matrix $H = 2A$

$$H = \begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix}$$

3. $\delta(x) = x_1^2 - 2x_2^2 + x_1x_2 - 2x_2x_3 + 5x_1x_3$

4. $\delta(x) = 3e^{2x_1+1} + 2e^{x_2+5}$

Check for quadratic form & write the Hessian matrix for the func.
 $3e^{2x_1+1} + 2e^{x_2+5}$

The above problem is not in the form of quadratic.

$\therefore$ we are ^{going to} apply the nature & definiteness of Hessian matrix

Note:- The relation $H=2A$ is true only for quadratic form & may not be true for other forms.

NATURE & DEFINITENESS

Nature of Hessian matrix:-

We can check the definiteness based on Hessian matrix H by using principle minors [Diagonal elements (D)]

→ Positive definiteness [All principle minors are positive]

$$D_1 > 0, D_2 > 0, D_3 > 0 \text{ \& so on}$$

→ Negative definiteness [All principle minors are -ve]

$$D_1 < 0, D_2 < 0, D_3 < 0$$

Alternating sign & the first principle minor should be

-ve

→ Positive semi definite

$$\text{i.e. } D_1 > 0, D_2 \geq 0, D_3 \geq 0 \text{ \& so on and atleast one of } D_i = 0$$

→ Negative semi definite

$$\text{i.e. } D_1 < 0, D_2 \leq 0, D_3 \leq 0 \dots \text{ and atleast one of } D_i = 0$$

where $D_1, D_2, D_3 \dots$ are called principle diagonal minors

MATRIX MINOR TEST

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here the size is 3×3 therefore we will be having 3 minors, they are

$$D_1 = |a_{11}|$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

16. Decide the definiteness of the function $f(x) = 3x_1^2 + 2x_2^2 - 3x_3^2 - 6x_1x_3 + 10x_1x_2 + 4x_2x_3$

Sol Let $A = \begin{bmatrix} -3 & 5 & -3 \\ 5 & 2 & 2 \\ -3 & 2 & -3 \end{bmatrix}$

Hessian matrix :- $H = 2A$

$$H = 2 \begin{bmatrix} -3 & 5 & -3 \\ 5 & 2 & 2 \\ -3 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 10 & -6 \\ 10 & 4 & 4 \\ -6 & 4 & -6 \end{bmatrix}$$

To find principle minors

$$D_1 = |-6| = -6 < 0$$

$$D_2 = \begin{vmatrix} -6 & 10 \\ 10 & 4 \end{vmatrix} = -24 - 100 = \underline{\underline{-124}} < 0.$$

$$D_3 = \begin{vmatrix} -6 & 10 & -6 \\ 10 & 4 & 4 \\ -6 & 4 & -6 \end{vmatrix} = -6(-24 - 16) + 10(-16 + 24) - 6(40 + 24) \\ = -6(-40) - 10(-36) - 6(64) \\ = 240 + 360 - 384 \\ = \underline{\underline{216}} > 0$$

$\therefore H$ is indefinite

7. $f(x) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100.$

8. The given problem is not in quadratic form $\therefore$ we go by H with P. Derivatives

$$\therefore H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}_{2 \times 2}$$

$$= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}_{3 \times 3}$$

$$\frac{\partial f}{\partial x_1} = 4x_1 + 10 \quad \frac{\partial f}{\partial x_2} = 2x_2 + 8 \quad \frac{\partial f}{\partial x_3} = 6x_3 + 6$$

$$\frac{\partial^2 f}{\partial x_1^2} = 4 \quad \frac{\partial^2 f}{\partial x_2^2} = 2 \quad \frac{\partial^2 f}{\partial x_3^2} = 6$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_1} = 0$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_2} = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_3} = 0$$

$$H = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$D_1 = |a_{11}| = |4| = 4 > 0$$

$$D_2 = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 - 0 = 8$$

$$D_3 = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 4(12-0) - 0(0-0) + 0(0-0) \\ = 48 - 0 - 0 \\ = \underline{\underline{48}} > 0$$

all principle minors are +ve.
 $\therefore$ Positive definite

18. $f(x) = 6x_1 + 8x_2 - x_1^2 - x_2^2$

Sol

$$\frac{\partial f}{\partial x_1} = -2x_1 + 6$$

$$\frac{\partial f}{\partial x_2} = -2x_2 + 8$$

$$\frac{\partial^2 f}{\partial x_1^2} = -2$$

$$\frac{\partial^2 f}{\partial x_2^2} = -2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_1} = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$D_1 = |-2| = -2$$

$$D_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 - 0 = \underline{\underline{4}}$$

$\therefore H$ is negative definite (first -ve then +ve)

19. Check the definiteness of the function $f(x) = x_1^3 - x_2^2 + 3x_2 + 10$ at the point $(0, 1.5)$ & $(2, 2)$.

 D_1
 D_2

$$\frac{\partial f}{\partial x_1} = 3x_1^2$$

$$\frac{\partial f}{\partial x_2} = -2x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1$$

$$\frac{\partial^2 f}{\partial x_2^2} = -2$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$$

$$H = \begin{bmatrix} 6x_1 & 0 \\ 0 & -2 \end{bmatrix}$$

At $(0, 1.5)$

$$D_1 = |6x_1| = 6x_1 = 6(0) = \underline{\underline{0}}$$

$$D_2 = \begin{vmatrix} 6x_1 & 0 \\ 0 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & -2 \end{vmatrix} = \underline{\underline{-2}}$$

$\therefore H$ is indefinite at $(0, 1.5)$

At (2, 2)

$$D_1 = |6x_1| = 6(2) = 12 > 0$$

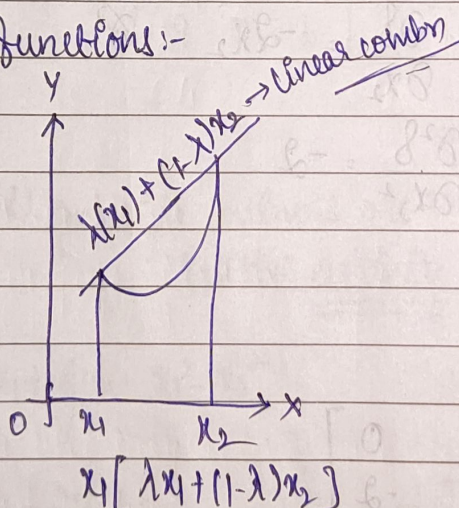
$$D_2 = \begin{vmatrix} 6x_1 & 0 \\ 0 & -2 \end{vmatrix} = -12x_1 = -12(2) = \underline{\underline{-24 < 0}}$$

$\therefore H$ is indefinite at (2, 2)

CONVEX OPTIMIZATIONS.

Definition:- Convex optimization is a subfield of mathematical optimization that studies the problem of minimizing the convex function on convex sets

Convex functions:-



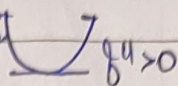
↳ convex linear combn of x_1, x_2

A function $f: S \rightarrow \mathbb{R}$ is said to be convex function if

$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ for any $x_1, x_2 \in S$ where S is a convex set and $\lambda \in [0, 1]$

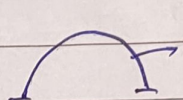
NOTE:-

→ convex means having a surface that is curved has rounded up
ex:- contact lens.

convex  $f'' > 0$

→ convex func always minimise the problem if $f'' \geq 0$ for all domain of x , then it is convex

→ concave function always maximise if $f'' \leq 0$ for all domain points, then it is concave

 concave

Case 1 for one dimension function, if $f'' \geq 0 \forall$ domain points, then it is convex

Ex:- $f(x) = x^2$

$f'(x) = 2x$

$\therefore$ it is convex

$f''(x) = 2 > 0$

if $f'' \leq 0 \forall$ domain points, then it is concave

Ex:- $f(x) = \log x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} < 0 \therefore$ it is concave.

Ex:- $f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f''(x) = -\frac{1}{4} x^{-3/2} < 0$

if $f'' < 0$ and $f'' \geq 0$ for some points of the domain, then it is neither concave nor convex

$$\text{Ex:- } f(x) = x \log x$$

$$f'(x) = x \left(\frac{1}{x} \right) + \log x$$

$$f''(x) = \frac{1}{x} < 0 \text{ if } x \text{ is } -ve \quad \therefore \text{It is neither convex nor concave}$$
$$> 0 \text{ if } x \text{ is } +ve$$

Case 2:- For higher dimension problem, we use Hessian matrix

Working Rule

→ Necessary conditions:

We diff the given func partially & equate to zero to get stationary points as x_1, x_2

$$\text{i.e. } \frac{\partial f}{\partial x_1} = 0 \quad \frac{\partial f}{\partial x_2} = 0 \dots$$

→ Sufficient conditions:

Compute Hessian matrix H for all stationary points obtained in step 1

NOTE The above necessary & sufficient cond is applicable only if it is dependent of x_1, x_2

* If H is independent of x_1, x_2 , then it is enough to apply to check for convex/concave by analysing the nature of Hessian matrix

Brief note on convexity & concavity

A func of a single variable is concave if every line segment joining 2 points on its graph does not lie above the graph at any point

Mathematically a func of a single variable is convex if every line segment joining 2 points on its graph does not lie below the graph at any point

Ex 1 $y = x^3 - 6x^2 + 12x$
 $y' = 3x^2 - 12x + 12$
 $y'' = 6x - 12$

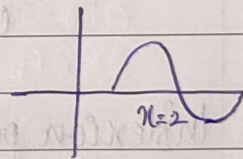
$$y''(x) = 0$$

$$\Rightarrow 6x - 12 = 0$$

$$6x = 12$$

$$\underline{\underline{x = 2}}$$

$y''(x)$ is -ve upto 2 and positive after that so concave downward upto $x = 2$ & then convex upward from $x = 2$
 Point of inflexions of the given function is at $x = 2$



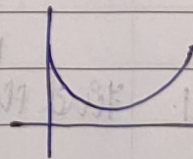
20 Discuss the convexity of $y = 3x^2 - 5x + 2$ with graph.

Sol. D. part x

$$y'(x) = 6x - 5$$

convex

$$\underline{\underline{y''(x) = 6 > 0}} \rightarrow \text{concave upward}$$



21 Discuss the func $f(x) = 2x^3 - 3x^2$. Find the interval on which f is concave up & down. Find the inflexion points of f .

Sol

$$f(x) = 2x^3 - 3x^2$$

$$f'(x) = 6x^2 - 6x$$

$$f''(x) = 12x - 6$$

Now consider

$$f''(x) = 0 \Rightarrow \text{Convex set}$$

$$12x - 6 = 0$$

$$12x = 6$$

$$x = \frac{6}{12}$$

$$\underline{x = 1/2}$$

Put $x=0$ in 2nd derivation

$$f''(x) = 12(0) - 6$$

$$= -6 < 0$$

↳ Concave downward

Put $x=1$ in 2nd der

$$f''(1) = 12(1) - 6$$

$$= 6 > 0$$

↳ convex upward

$\therefore f$ is concave downward from $(-\infty, 1/2)$ &
convex upward from $(1/2, \infty)$

Inflexion point can be calculated by putting $x = 1/2$ in $f(x)$

$$f(1/2) = 2(1/2)^3 - 3(1/2)^2$$

$$= -1/2$$

$\therefore$ The inflexion point is $\underline{\underline{(1/2, -1/2)}}$

minimize (convexity)

21. Find the max/min of the sum $f(x) = x_1^2 + x_2^2 + x_1 + x_2 - 1$

$$\frac{\partial f}{\partial x_1} = 2x_1 + 1$$

$$\frac{\partial f}{\partial x_2} = 2x_2 + 1$$

$$\frac{\partial^2 f}{\partial x_1^2} = 2$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 2$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_1} = 0$$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\therefore H$ is independent of x_1, x_2 , it is enough to check the nature of H for convex

$$D_1 = |2| = 2 > 0$$

$$D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = \underline{\underline{4}} > 0.$$

$\therefore$ Here all the principle minors are positive

$\therefore$ It is strictly convex & +ve definite

$\therefore$ The function is minimised

Q2. Optimise the func for $2x_1^2 - x_2^2 + x_1x_2 + 3x_1 + 4x_2 + 17$

$$\frac{\partial f}{\partial x_1} = 4x_1 + x_2 + 3 \quad \frac{\partial f}{\partial x_2} = -2x_2 + x_1 + 4$$

$$\frac{\partial^2 f}{\partial x_1^2} = 4 \quad \frac{\partial^2 f}{\partial x_2^2} = -2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 1 \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$$

$$H = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$$

$$D_1 = |4| = 4 > 0$$

$$D_2 = \begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} = -8 - 1 = \underline{\underline{-9}} < 0$$

$\therefore$ It is neither concave nor convex since the principle minors are of alternate sign but the first minor is +ve. Thus, they give neither maximum nor minimum

23 $4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 + 4$ $H = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}$ $-4 < 0$
 $12 > 0$

Sol Since the principle minors are alternating signs, & first minor is -ve, then it is -ve definiteness
Hence $f(x)$ is strictly concave
 $\therefore$ It will give maximum

24. Find the maximum or minimum of $f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$

Sol $\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1$ $\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2$

$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4$ $\frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8$

$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$

$H = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix}$ independent of x_1, x_2

Necessary cond

$\frac{\partial f}{\partial x_1} = 0$ $\frac{\partial f}{\partial x_2} = 0$

$3x_1^2 + 4x_1 = 0$

$x_1(3x_1 + 4) = 0$

$x_1 \neq 0 \implies 3x_1 = -4$

$x_1 = -\frac{4}{3}$

3

D_1 -ve D_2 -ve
Both +ve min
other -ve

DATE

$$6x_2 + 8 = 0 \quad 3x_2^2 + 8x_2 = 0$$

$$x_2 (3x_2 + 8) = 0$$

$$x_2 = 0 \quad 3x_2 + 8 = 0$$

$$x_2 = -\frac{8}{3}$$

$\therefore$ The stationary points are
(0,0) (0, -8/3) (-4/3, 0) (-4/3, -8/3)

Sufficient conditions

$$D_1 = |6x_1 + 4|$$

$$D_2 = \begin{vmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{vmatrix} = (6x_1 + 4)(6x_2 + 8)$$

*

Point (x_1, x_2)	Value of D_1	Value of D_2	Nature of $f(x)$ at x	Nature of H
(0,0)	$4 > 0$	$32 > 0$	Minimum	6 +ve definite
(0, -8/3)	$4 > 0$	$-32 < 0$	Saddle point	418/27 Indefinite
(-4/3, 0)	$-4 < 0$	$-32 < 0$	Saddle point	144/27 Indefinite
(-4/3, -8/3)	$-4 < 0$	$32 > 0$	Maximum	50/3 -ve definite

$$x \log x$$

$$\begin{aligned} f(x) &= \frac{x \log_e x}{\log_e^2} = \frac{1}{\log_e^2} \left[x \cdot \frac{1}{x} + \log x \right] \\ &= \frac{1}{\log_e^2} + \frac{\log x}{\log_e^2} \\ &= \frac{1}{\log_e^2} + \frac{1}{x} \end{aligned}$$